Coupled Pendulums

Equipment 2 Rotary Motion sensors mounted on a horizontal rod, 2 vertical rods to hold horizontal rod, 2 bench clamps to hold the 2 vertical rods, 2 rod clamps to hold horizontal rod to vertical rods, 2 rods and 2 masses from mini-rotational accessory, size 18 rubber band

1 Introduction

The motion of 2 coupled identical pendulums will be studied. The deflection angles will be assumed small enough so that the equations of motion can be linearized. The coupling will be assumed weak. This means that the force between the two pendulums is weak compared to the force of gravity on each pendulum. It is assumed that each pendulum oscillates in a plane and has one degree of freedom. The system of both pendulums has 2 degrees of freedom. While a general motion of this system might seem complicated, any motion of this system can be described by the sum of two normal modes. In a given normal mode all parts of the system oscillate at the same frequency and pass through their equilibrium positions at the same time. The two normal modes of this system will be observed and the frequency of each mode measured.

A particularly fascinating aspect of this system is that if the two pendulums are identical and the initial conditions chosen appropriately, the energy passes back and forth between the two pendulums. This feature will be examined and is known as "beats." If the pendulums are not identical, the transfer of energy is not complete.

The idea of normal modes in linearized systems is very useful and can be extended to systems of many degrees of freedom. The phenomena of beats occurs in many diverse fields ranging from mechanics to quantum mechanics.

2 Apparatus

The apparatus is shown in Figure 1. Two rotary motion sensors are mounted on a horizontal rod. The pulley of each sensor has a rod with a moveable weight attached to it to make the sensor a pendulum. A rubber band connecting the two rods couples the two pendulums.

3 Theory

We obtain the equations of motion for the system of two pendulums. The pendulums are designated a and b. The following assumptions and approximations are made.

- The two pendulums are identical and when uncoupled have the same natural frequency.
- The angular displacements of the pendulums from vertical are given by θ_a and θ_b. If the pendulums are displaced in the same direction both these angles have the same sign.
- The pendulums are simple. The pendulum rods are massless and are of length ℓ. The masses, denoted by M, are points.
- The deflection angles are small and the equations of motion are linearized: $\sin\theta \cong \theta$.

- Let K be a constant. The force on pendulum a in the direction of motion due to the coupling is $-K\ell(\theta_a - \theta_b)$ and the force on pendulum b due to the coupling is $-K\ell(\theta_b - \theta_a)$. (The form $K\ell$ is chosen for notational convenience.)

The equations of motion become

$$\frac{d^2}{dt^2}\theta_a + \frac{g}{\ell}\theta_a + \frac{K}{M}(\theta_a - \theta_b) = 0, \qquad \textit{and} \tag{1}$$

$$\frac{d^2}{dt^2}\theta_b + \frac{g}{\ell}\theta_b + \frac{K}{M}(\theta_b - \theta_a) = 0. \tag{2}$$

We assume that it is possible to find values of ω such that

$$\theta_a = A\cos(\omega t + \phi) \qquad \textit{and} \tag{3}$$

$$\theta_b = B\cos(\omega t + \phi) \tag{4}$$

are solutions to Eqs. 1-2. The angular frequency of the motion is ω. A, B, and ϕ are constants which can be chosen to satisfy initial conditions. Substituting Eqs. 3-4 into Eqs. 1-2 yields the following algebraic equations.

$$\left(\frac{g}{\ell} + \frac{K}{M} - \omega^2\right)A - \frac{K}{M}B = 0, \quad \textit{and} \tag{5}$$

$$-\frac{K}{M}A + \left(\frac{g}{\ell} + \frac{K}{M} - \omega^2\right)B = 0. \tag{6}$$

These two homogeneous equations have solutions only for the following two values of ω:

$$\omega_1^2 = \frac{g}{\ell} \quad \textit{and} \quad \omega_2^2 = \frac{g}{\ell} + \frac{2K}{M}. \tag{7}$$

For ω_1, or mode 1, either Eq. 5 or Eq. 6 yields $A_1 = B_1$. The subscripts on A and B indicate that these are the amplitudes associated with mode 1. This is a symmetric mode where the pendulums have the same amplitude and the same phase. They vibrate together.

For ω_2, or mode 2, either Eq. 5 or Eq. 6 yields $A_2 = -B_2$. The subscripts on A and B indicate that these are the amplitudes associated with mode 2. This is an antisymmetric mode where the pendulums have the same amplitude but opposite phases.

The motions for mode 1 are given by

$$\theta_{a1} = A_1\cos(\omega_1 t + \phi_1) \quad \textit{and} \quad \theta_{b1} = A_1\cos(\omega_1 t + \phi_1). \tag{8}$$

The motions for mode 2 are given by

$$\theta_{a2} = A_2\cos(\omega_2 t + \phi_2) \quad \textit{and} \quad \theta_{b2} = -A_2\cos(\omega_2 t + \phi_2). \tag{9}$$

Any motion of the linearized system is given by

$$\theta_a = \theta_{a1} + \theta_{a2} = A_1\cos(\omega_1 t + \phi_1) + A_2\cos(\omega_2 t + \phi_2), \quad \textit{and} \tag{10}$$

$$\theta_b = \theta_{b1} + \theta_{b2} = A_1\cos(\omega_1 t + \phi_1) - A_2\cos(\omega_2 t + \phi_2). \tag{11}$$

The four constants A_1, A_2, ϕ_1, and ϕ_2 are chosen to satisfy the initial conditions.

4 Beats

Unless this system is in a single mode, some energy is transferred back and forth between the modes at a frequency called the beat frequency. The energy transfers are complete when the two modes are present in equal amounts.To achieve this take $A_1 = A_2 = A$. Also take $\phi_1 = \phi_2 = 0$. For these particular initial conditions the motions become

$$\theta_a = A\cos\omega_1 t + A\cos\omega_2 t \quad and \tag{12}$$

$$\theta_b = A\cos\omega_1 t - A\cos\omega_2 t. \tag{13}$$

Using standard trigonometric formulae, Eqs. 12-13 become

$$\theta_a = 2A\cos\left(\frac{\omega_2 - \omega_1}{2}\right)\cos\left(\frac{\omega_1 + \omega_2}{2}\right) \quad and \tag{14}$$

$$\theta_b = 2A\sin\left(\frac{\omega_2 - \omega_1}{2}\right)\sin\left(\frac{\omega_1 + \omega_2}{2}\right). \tag{15}$$

Defining the beat frequency ω_b and the average frequency ω_{av} as

$$\omega_b = \frac{\omega_2 - \omega_1}{2} \quad and \quad \omega_{av} = \frac{\omega_1 + \omega_2}{2}, \tag{16}$$

Eqs. 14-15 can be written as

$$\theta_a = [2A\cos\omega_b t]\cos\omega_{av} t \quad and \quad \theta_b = [2A\sin\omega_b t]\sin\omega_{av} t. \tag{17}$$

The weak coupling between the pendulums results in $\omega_b \ll \omega_{av}$. In Eq. 17 the two quantities in square brackets act as slowly varying amplitudes for the faster varying $\cos\omega_{av} t$ and $\sin\omega_{av} t$ factors. Due to the phase difference between these two amplitudes, one pendulum has maximum amplitude when the other is at rest. The energy goes back and forth at the beat frequency.

5 Setting Up The Apparatus

- Check that the height of the horizontal rod holding the two rotary motion sensors is such that the pendulum rods just clear the lab bench by one or two cm. Check that the bench clamps and rod clamps are tight but do not tighten excessively. The reason for the above precautions is to prevent loose mounting from coupling the pendulums.

- Plug the left rotary motion sensor in channels 1 and 2, with the yellow plug in channel 1. Plug the right rotary motion sensor in channels 3 and 4, with the yellow plug in channel 4. This "reversal" of the plugs should give positive signals when the two pendulums are deflected in the same direction. If you find this is not the case when you start to take data, reverse the plugs in channels 3 and 4. (It turns out that the wiring of all the rotary motion sensors is not identical.)

- Program the interface for the two rotary motions sensors, choosing 1440 divisions per rotation for both.

- Click the sampling options button in the left experiment setup window and set the sampling rate to 100 Hz.
- Set the two masses at about 16.5 cm below the two pivot points.
- Put the rubber band at about 9.5 cm below the pivot points. Adjust the distance between the pendulums so that the rubber band has very little tension. There should be just enough tension to keep the rubber band in place when the two pendulums are swinging.
- Open the graph display by dragging its icon to the icon of the rotary motion sensor below channels 1 and 2. icon. Choose angular position. Click the add-a-plot menu button on the graph display and choose angular position for the second rotary motion sensor. Click the display options button on the graph display and remove the check from the data points box by clicking the box. You will probably find the display easier to read without the data points.
- Remove the rubber band. With the pendulums at rest click REC and set the left pendulum in motion with modest amplitude, leaving the right pendulum and rest. Let it oscillate 4 or 5 times and click STOP. Fit a sine curve to an oscillation or two and determine the frequency. CAUTION. Be sure you get a good fit. If you don't, try again until you do. In a good fit the curve will lie right on your data and the chi^2 will be 10^{-2} or less. Question. What are the units of the frequency given by SWS for the sine wave fit?
- Repeat for the right pendulum. Adjust the position of the mass on the right pendulum until its frerquency is as close to that of the left pendulum as you can make it. Both pendulums should now be identical. The common frequency for both pendulums is the normal mode frequency for the symmetric mode. Replace the rubber band. Question. With the rubber band in place, why is the frequency of the symmetric mode (both pendulums swinging in phase with the same amplitude) identical to the common frequency of the two pendulums when the rubber band is removed?

6 Observing Beats And Adjusting The Coupling

With the rubber band at about 9.5 cm below the pivot points and both pendulums at rest, click REC. Start one pendulum swinging. This excites both modes equally as discussed in the section on Beats. Observe the system go through a few "beats" and click STOP. Compare your data to the predictions of Eqs. 17. Measure the time between amplitude minimums, and count the cycles of one pendulum that carries it from one minimum in its amplitude to the next minimum in its amplitude. Adjust the rubber band up and down and observe and measure the difference in the beat pattern. Moving the rubber band up and toward the pivot point will decrease the coupling. Moving the rubber down will increase the coupling. Why? Discuss how changing the coupling changes the beat pattern. Finally, adjust the rubber band so that each pendulum has about 10 to 12 oscillations between successive amplitude minimums. Be careful not to move the rubber band after this adjustment.

7 The Antisymmetric Mode

In this section the pendulums will be set oscillating in the antisymmetric mode with as little of the symmetric mode present as possible. With both pendulums at rest click REC. Move one pendulum to one side and hold it and move the other pendulum an equal amount to the other side and hold it. Observe the graph display and make the magnitudes of the deflections as equal as possible. Let go of both pendulums at the same time. If you have succeeded in making a pure antisymmetric mode there will be no beats. This is probably not possible, but do the best you can. Practice a bit. When you are satisfied, measure the frequency of the antisymmetric mode.

The beat frequency is given by Eq. 16 in terms of the two normal mode frequencies which you have now measured. How does the calculated beat frequency compare to your experimental result?

8 Who is driving Whom?

When both modes are excited equally and each pendulum alternately oscillates and stops, there are moments when both pendulums have about the same amplitude. How is it that one pendulum is decreasing in amplitude and the other is increasing in amplitude? Expand a portion of your graphs where the amplitudes are about equal and look at the phase between the pendulums. Note which pendulum amplitude is getting larger. Discuss your observations.

9 Arbitrary Initial Conditions

Starting one pendulum oscillating while the other is at rest is a very special initial condition that excites both modes equally. Try a few other initial conditions and see what happens. Describe your results.

10 The Two Pendulums Not Identical

Lower one of the pendulums by about 2 cm. Start one pendulum swinging and observe the results. Discuss your observations.

11 Inadvertent Coupling

Move the mass up to about where it was and remove the rubber band. Start one pendulum swinging. Does anything happen? Now move the horizontal bar that holds the two pendulums up to the top of the vertical rods. Start one pendulum swinging and see if anything happens. Discuss your results.

12 Finishing Up

Please leave the bench as you found it. Thank you.

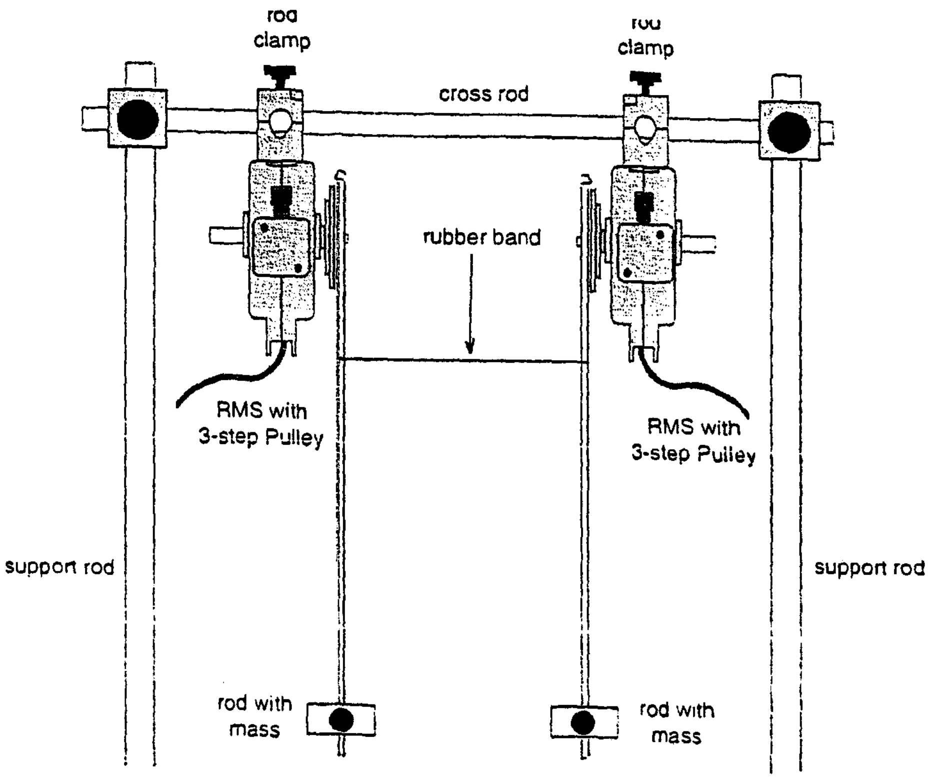
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Fig. 1