Rotational Motion

1 Purpose

- To investigate the relationship between torque, moment of inertia, and angular acceleration for a rigid body rotating about a fixed axis. Conservation of angular momentum will also be examined.

- To do a straight line fit and determine the uncertainties on the fit slope.

2 Theory

2.1 Equation of Motion for a Rotating Rigid Body

The equation of motion for a particle moving in a circle may be derived as follows. Consider a net force $\mathbf{F}$ acting on a point mass m constrained to move in a circle of radius r. If $\mathbf{F}$ is only in the radial direction, then the speed of the mass will be constant, resulting in what is called "uniform circular motion". If, however, the force has a component along the direction of motion, m may speed up or slow down. The dynamics of this are given by Newton's 2nd Law $\mathbf{F} = m\mathbf{a}$, where $\mathbf{a}$ is the acceleration. At any instant of time it is convenient to write the component of this equation which is in the direction of the velocity, that is, tangential to the circle at the point where the mass is. Denote the component of the force in that direction as F_{tan}. For circular motion it is also the case that the component of the acceleration tangential to the circle is $r\alpha$, where r is the radius of the circle on which the particle moves and α is the angular acceleration in rad/s^2. The relevant component of Newton's 2nd Law becomes $F_{tan} = mr\alpha$. Multiplying both sides of this equation by r gives $rF_{tan} = mr^2\alpha$. The quantity rF_{tan} is called the torque τ about the axis of the circle. One could also say that this is the component of the torque along the axis of rotation. The quantity mr^2 is called the moment of inertia I of the mass about the axis of the circle. (The axis of the circle is a line that runs through the center of the circle and is perpendicular to the plane of the circle.) The equation of motion now has the form $\tau = I\alpha$.

This analysis is easily extended to a rigid body rotating about an axis fixed in space. Suppose the rigid body to be constructed of point masses m_i each moving in a circle of radius $r_{i\perp}$ and acted upon by a force $\mathbf{F_i}$ whose component in the direction of the velocity is $F_{i,tan}$. The $r_{i\perp}$ differ for every i, as each represents the perpendicular distance from the axis of rotation to m_i. The velocity and acceleration will also in general be different for every mass i. However, all points will have the same angular acceleration α. The equation of motion for each mass is $r_{i\perp}F_{i,tan} = m_i r_{i\perp}^2\alpha$. Summing this over all the point masses of the rigid body we have

$$\sum_i r_{i\perp}F_{i,tan} = \sum_i m_i r_{i\perp}^2\alpha.$$

The forces acting on any mass m_i can be divided into internal and external ones. Internal forces are those exerted on m_i by the other masses of the rigid body. External forces are all other forces that originate outside of the system of masses. It can be shown that due to Newton's 3rd Law the torques due to the internal forces of the rigid body will cancel. Without changing notation, we assume the torque is due solely to external forces. Extending our definitions of torque and moment of inertia to $\tau = \sum_i r_{i\perp} F_{i,tan}$ and $I = \sum_i m_i r_{i\perp}^2$, the equation of motion for a rigid body rotating about a fixed axis is $\tau = I\alpha$, where τ is the external torque. In words, for a rigid body rotating about a fixed axis, the sum of the external torques computed about that axis is equal to the moment of inertia computed about the same axis times the angular acceleration.

If the rigid body is described by a continuous mass distribution with a mass per unit volume of $\rho(\mathbf{r})$, the moment of inertia I is give by $\int \rho(\mathbf{r}) r_\perp^2 dv$, where dv is a differential volume element and $r_\perp$ is the perpendicular distance from the axis to dv.

2.2 Torque and Angular Momentum

In the previous section we dealt only with one component of torque. More generally, the torque, $\boldsymbol{\tau}$, about some point $\mathbf{Q}$ exerted by a force $\mathbf{F}$ acting at a position $\mathbf{r}$ with respect to $\mathbf{Q}$ is given by $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$. Applying this definition to our rigid body rotation, if $\mathbf{Q}$ is taken anywhere along the axis of rotation, then the torque as it was defined in the previous section is just the component of torque along the axis of rotation.

Angular momentum $\mathbf{L}$ about the point $\mathbf{Q}$ for a particle with linear momentum $\mathbf{p}$ is defined as $\mathbf{L} = \mathbf{r} \times \mathbf{p}$. For a system of particles, if $\boldsymbol{\tau}$ represents the sum of the external torques on the object about a point, and $\mathbf{L}$ represents the sum of the angular momenta for the particles about the same point, it can be shown that an equation of motion is $\boldsymbol{\tau} = \frac{d}{dt}\mathbf{L}$. If any component of $\boldsymbol{\tau}$ is zero, that component of the angular momentum is a constant.

2.3 Angular Momentum and Rigid Body Rotation

For a single rigid body rotating with angular velocity ω about an axis, it can be shown that the angular momentum along the axis is given by $L = I\omega$. The equation of motion then becomes $\tau = d/dt(I\omega) = I\alpha$, as I is a constant. In this case, if τ is equal to zero, than ω is a constant. It is interesting to consider a body that is rigid, then deforms itself, and then becomes rigid again. In the deformation, assume that the moment of inertia is changed but that there are no external torques. The angular momentum $I\omega$ must stay the same, so that for example if I is decreased then ω must increase. An ice-skater pulls in his/her arms to increase the rate of spin.

Another case is that of two rigid bodies, one of which is spinning about an axis and has angular momentum $I\omega$ and the other is not spinning and has no angular momentum. Assume no external torques. If one of these objects is dropped on the other and they stick, the angular momentum must stay the same. We have $I\omega = I_f \omega_f$, where I_f is the final moment of inertial of the composite body and ω_f is the final angular velocity.

2.4 Rotational Kinetic Energy

The kinetic energy of a rigid body rotating about an axis is $\frac{1}{2}I\omega^2$.

2.5 Moments of Inertia

In these experiments various rigid bodies will be rotated about an axis. Some of these rigid bodies will be made of components. The moment of inertia of each component about the chosen axis can be calculated and then the total moment of inertia obtained by summing the moments of inertia of the components. We list the moments of inertia of various components each of mass M and assumed uniform. All of these can be calculated using the definition of the moment of inertia, $I = \int \rho(\mathbf{r}) r_\perp^2 dv$

1. A small mass a distance R from the axis: $I = MR^2$.
2. A thin rod of length L about an axis through the center: $I = \frac{1}{12}ML^2$.
3. A solid cylinder (or disk) of radius R about its axis: $I = \frac{1}{2}MR^2$.
4. A tube of outer radius R_2 and inner radius R_1 about its axis: $I = \frac{1}{2}M(R_1^2 + R_2^2)$.
5. A tube of outer radius R_2 and inner radius R_1 and length L about an axis through the center of the tube and perpendicular to the tube axis: $\frac{1}{12}ML^2 + \frac{1}{4}M(R_2^2 + R_1^2)$.
6. **The parallal axis theorem** The moment of inertia of a rigid body of mass M about any axis $\mathbf{A}$ is equal to $I_{CM} + Md^2$, where I_{CM} is the moment of inertia of the body about an axis parallal to $\mathbf{A}$ but going through the center of mass, and d is the distance of the center of mass to the axis $\mathbf{A}$.

2.6 Equation for the Experiments

A horizontal pulley is mounted on a vertical axis (see Fig. 1). Let the radius of the pulley used be R_p. A rigid body whose moment of inertia about the pulley axis is I is mounted on the pulley. A string is wound around the pulley, goes horizontally to a “super” pulley, and then goes straight down to a hanging mass m. Let the tension in the string be T. The equations of motion for the rotating rigid body and the hanging mass are $\tau = R_pT = I\alpha$ and $mg - T = ma = mR_p\alpha$, where a is the linear acceleration of m, α is the angular acceleration of the rigid body, and g is the acceleration of gravity. The tension T can be eliminated to give

$$\alpha = \frac{mR_pg}{mR_p^2 + I}.$$

Use this equation in your analyses. In this analysis, the moments of inertia of the two pulleys are taken to be zero.

3 Equipment

Equipment: Capstone, rotary motion sensor with a “super” pulley mounted on 80 cm rod and heavy duty bench clamp (PASCO ME-9472), horizontally mounted 3-step (that is, 3 different radii available) pulley, string with loop at one end and small white bead at the other end (125 cm bead to end of loop), rod with 2 brass weights, disk, cylinder, set of weights with hooks, vernier calipers, meter stick.

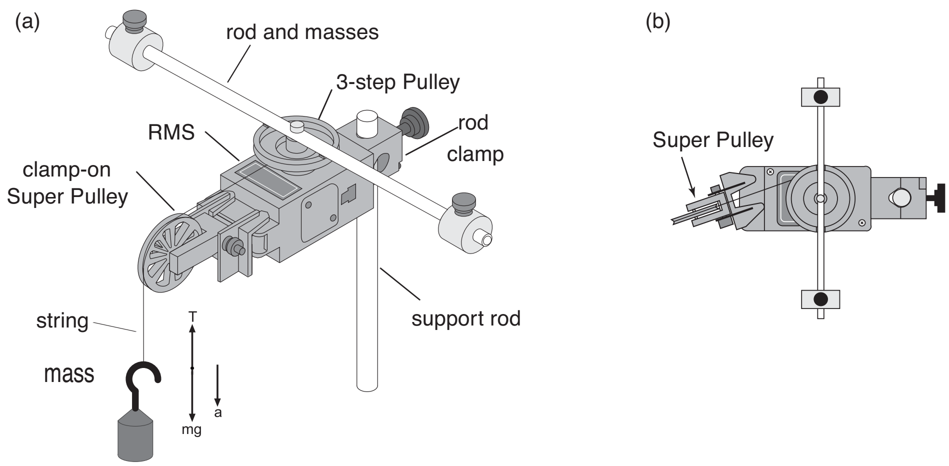


Figure 1: Rotary motion sensor. (a) Side view. (b) Top view.

Rotary Motion Sensor: A 2 plug Capstone digital sensor (see Fig. 1). It is similar to the smart pulley in that it measures the angular rotation of a pulley, but differs in that the rotary motion sensor can detect the direction of the rotation. There are two angular resolutions: 360 divisions per rotation (1°) and 1440 divisions per rotation (1/4°). The plugs should be inserted into 2 adjacent digital channels. Reversing the order of the plugs will change the sign of the sensor output.

4 General Procedures

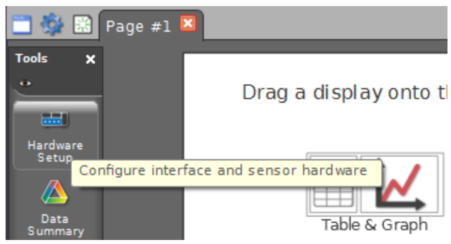
4.1 Use of the equipment

The rotary motion sensor (RMS) is mounted on a vertical support rod as shown in Fig. 1. A 3-step pulley is mounted on the axis of the RMS. Mount the rod with 2 brass weights on the 3-step pulley as shown in Fig. 1. Make sure to fix it in place with the screw provided. A loop in one end of the string can be attached to a hanging mass. The other end of the string has a small bead on it. This end of the string can be attached to one of the three grooves in the pulley by catching the bead in one of the notches in the rims of the pulleys. The height of the super pulley should be adjusted so that the string between the two pulleys is horizontal. The angle of the super pulley should be adjusted so that the string enters the super pulley parallel to the groove of the super pulley. The super pulley can be easily moved by loosening the two screws on the bracket of the super pulley. The string can be wound onto one step of the 3-step pulley by rotating the rod until the hanging mass is just below the super pulley.

A second series experiments use a disk mounted on the 3-step pulley instead of a rod. Remove the rod by rotating the screw holding the rod to the 3-step pulley. Do not rotate this screw so much that you remove the screw from the rod. The screw is “captured” and is designed to stay on the rod. Remove the 3-step pulley by pulling it off. Observe that the hole in the pulley has a projection that fits into a slot in the shaft of the RMS. Turn the pulley over so the small pulley is on top and put it back onto the axle, aligning the projection in the hole of the pulley with the slot in the RMS shaft. The end of the RMS pulley that now shows is square. Mount the disk so that the square end of the pulley goes into the square hole in the disk. Use the screw in the dish on the bench to attach the disk to the pulley. For another experiment a hollow cylinder is mounted on the disk. Note that the cylinder has 2 projections that fit into holes on the disk. When you remove the disk from the pulley put the screw back into the dish.

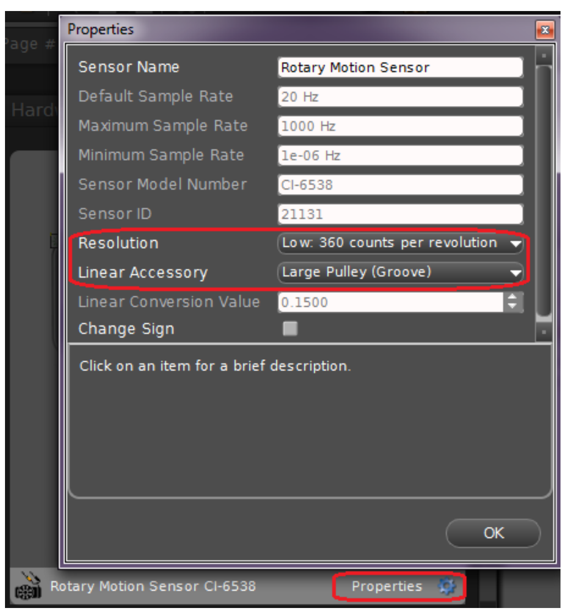
4.2 Programming and saving data

Go to the desktop screen and double click on the PASCO Capstone software icon. The Capstone software will open up and you should see a white screen labeled Page and a tools column. In the tools column click on the **Hardware Setup**.



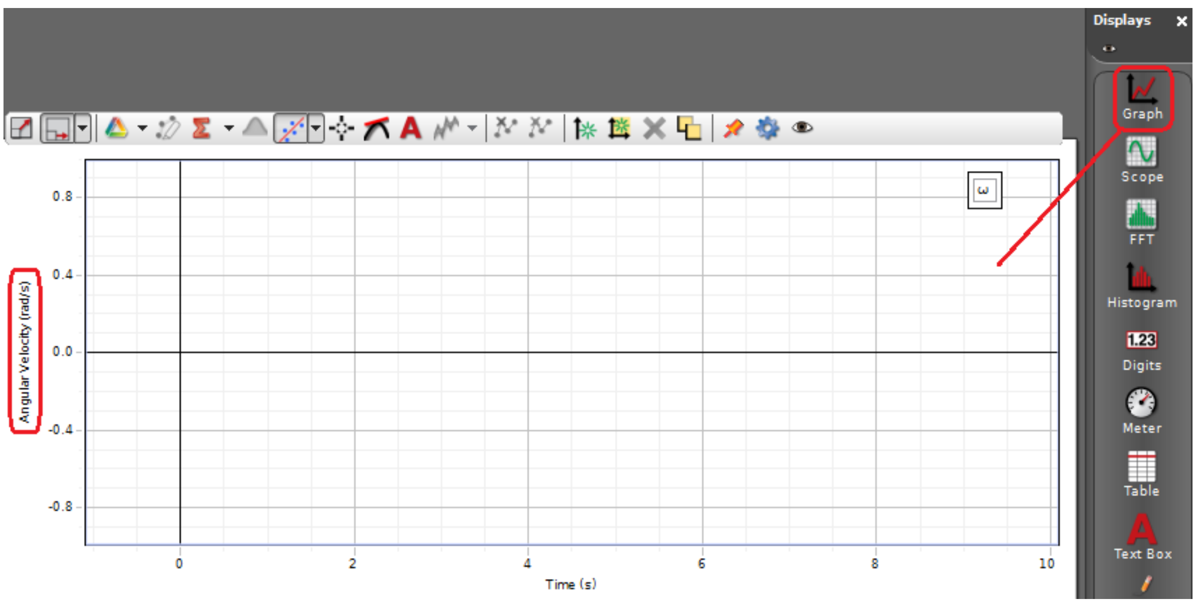
An image of the interface will pop up in the hardware setup window. On the interface click on the digitial channel that the rotary motion sensor is plugged into. A window will pop up with a selection of different sensors. Scroll down and select the rotary motion sensor.

To program the digital rotary motion sensor, click on properties in the lower part of the hardware setup window. A properties window will pop up. Go to Resolution menu, choose 360 counts per revolution and select the appropriate Large Pulley from the LinearAccesory menu. Click ok.



Next click on the tack located on the top right corner of the hardware setup window. Clicking on the tack prevents overlapping of the windows.

To create a graph display, go to the 'Displays' window on the right side. The graph icon is located in the display column. Find the 'graph icon. Drag and drop the graph icon from the "Displays" window to the white screen located next to the hardware setup window. A graph will pop up. Look at the graph and click on **select measurments** in the vertical axis. Choose **Angular Velocity (rad/s)** for the vertical axis. Keep the horizontal axis as time.



The slope of the angular velocity vs time line is the angular acceleration α. **Note: (In section 5 of the experiment is when you're to find angular acceleration)** You determine the slope by fitting a straight line through an appropriate portion of your data and use this slope to do preliminary analysis of your data while you are still in the lab.

For your lab report, however, you will need to save your data to a text file so that you can do a more thorough analysis later. To save your data as a text file, do the following. First need to make a table of your data by dragging the Table icon over to the center of the white data window. Under select measurements choose Angular Velocity, angVel (rad/s). There are two options to export data. First, you can save all the data by selecting the `Export Data` button under the `File` menu. Second, you can export selected data by using the columns in the data table. (Note: you can switch runs by clicking on the Run # on the top of the column in the data table and then select the data run you want.) In the data table left click on the Angular Velocity column. This selects all the data in that column. Next, right click on the data of the Angular Velocity column and select copy values. After, open the Notepad program and pasta the data. Also, you can use Excel. Save the data as `angularvelocity.txt`. E-mail the data files to yourself and to your lab mates so that you have multiple copies for later use.

5 Experiments

5.1 Rod & 2 Brass Cylinders

Measure the mass of the rod, the masses of the 2 brass cylinders, and the diameter of the large and medium grooves of the pulley. Mount the 3-step pulley, with the large pulley on top, and then the rod on the pulley laying the rod between the ridges on the face of the large pulley and using the captured screw. Put the 2 cylinders on the very ends of the rod. Measure the length of the rod and the distance from the axis to the middle of each cylinder. Determine α experimentally using a 20 g mass. Wind the string until the hanging mass is just below the super pulley. Let go of the rod and click Record. Just before the mass hits the floor click STOP and bring the rotation to rest by holding the shaft on the bottom side of the RMS with your fingers.

Take several runs and then repeat for a 30 g or 40 g mass. Compare your measurement to the theoretical value of α. You may treat the cylinders as point masses. Move the cylinders in so that they are outside the super pulley but just clear it when the rod is rotated. Repeat your measurements.

After doing the other experiments, if you have time repeat some of these experiments using the medium pulley for the string.

Store the data files and email them to to each member of your team.

5.2 Analysis and Questions for Rod & 2 Brass Cylinders Experiment

1. Using your measurements, calculate the expected value of the angular acceleration and the uncertainty in the value.

2. While in the lab, fit velocity versus time to determine the angular acceleration.

3. Determine the uncertainties in each quantity measured.

4. What error is introduced by treating the cylinders as point masses?

5. Write a Python program that fits your data to a straight line and determines the uncertainty in the angular acceleration. Is your determination consistent within errors with what your calculation predicts?

5.3 Disk and Cylinder

Measure and weigh the aluminum disk and the large cylinder. Remove the rod and the 3-step pulley. Remount the 3-step pulley with the small pulley on top, and then mount the disk on the 3-step pulley and secure it with a screw. Mount the large cylinder on the disk using the projections on the end of the cylinder and the holes in the disk.

Find α when a 20 g mass is used to exert a torque. Use this to determine I for the disk and cylinder system.

Now mount the cylinder on the disk with the axis of the cylinder horizontal. A hole in the side of the cylinder allows the cylinder and disk to be attached to the disk with the long screw. Using a 20 g mass, measure α to determine the moment of inertia of the system when the rotation is about an axis through the center of the cyinder but perpendicular the cylinder's axis.

5.4 Analysis and Questions for Disk and Cylinder Experiments

1. Determine the angular acceleration and its uncertainty for each of the situations studied: a disk rotating about its axis, a cylinder rotating about its axis, and a cylinder rotating about an axis perpendicular to the cylinder axis.

2. Determine the moment of inertia and its uncertainty for each of these cases.

3. Using your measurements, calculate the moment of inertia and its uncertainty for each of these cases.

4. Compare the measured and calculated values of the moment of inertia for each of these cases.

5.5 Conservation of Angular Momentum

Remove the string from the pulley. Remove the cylinder from the disk, leaving the disk connected to the pulley, then click Record, and set the disk spinning. This is best done by twirling the part of the axle of the RMS that is underneath the RMS between thumb and index finger. With the cylinder projections up (cylinder axis vertical), gently drop the cylinder as near the center of the disk as you can and press STOP. It will be difficult to drop the cylinder exactly onto the center of the disk. Measure the eccentricity of the dropped cylinder and use the parallel axis theorem to get the moment of inertia of the slightly eccentric cylinder.

5.6 Analysis and Questions for Conservation of Angular Momentum Experiment

1. Are your measurements consistent within uncertainties with conservation of angular momentum?

2. Are your measurements consistent within uncertainties with conservation of rotational kinetic energy?

6 Finishing Up

Please leave the bench as you found it. Be sure the second screw is in the dish. Thank you.