Conservation of Energy

1 Purpose

To study conservation of energy in the case of conservative forces acting on a system. Three mechanical systems are studied: a falling mass, a pendulum, and a mass oscillating on a spring.

2 Theory

2.1 General discussion of conservation of energy

If an object of mass m moves from an initial position $\mathbf{r}_i$ to a final position $\mathbf{r}_f$ under the influence of a force $\mathbf{F}$, the work done by the force on the object is given by

$$\int_{\mathbf{r}_i}^{\mathbf{r}_f} \mathbf{F} \cdot d\mathbf{r} = \left(\frac{1}{2}mv^2\right)_f - \left(\frac{1}{2}mv^2\right)_i \ . \tag{1}$$

Equation 1 is obtained by using Newton's 2nd law to replace $\mathbf{F}$ by $m\mathbf{a}$ and then integrating over the path (*i.e.* trajectory) of the object. Equation 1 is a completely general result, which explicitly depends on the path taken by the particle. However, if for some force the result of doing the integral is *independent of the path* taken by the object, that force is said to be a *conservative* force. Some examples of conservative forces are the gravitational force, the ideal spring force, and the electrical force.

The quantity appearing in each of the terms on the right hand side of Eq. 1 is called the *kinetic energy*

$$K = \frac{1}{2}mv^2 \ . \tag{2}$$

Since for a conservative force the integral in Eq. 1 is independent of path, it can be split into the sum of two integrals, one that goes from the starting point $\mathbf{r}_i$ to some arbitrary reference point $\mathbf{r}_0$, and another that goes from $\mathbf{r}_0$ to $\mathbf{r}_f$:

$$\int_{\mathbf{r}_i}^{\mathbf{r}_f} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathbf{r}_i}^{\mathbf{r}_0} \mathbf{F} \cdot d\mathbf{r} + \int_{\mathbf{r}_0}^{\mathbf{r}_f} \mathbf{F} \cdot d\mathbf{r} \tag{3}$$

$$= \int_{\mathbf{r}_0}^{\mathbf{r}_f} \mathbf{F} \cdot d\mathbf{r} - \int_{\mathbf{r}_0}^{\mathbf{r}_i} \mathbf{F} \cdot d\mathbf{r} \ , \tag{4}$$

To obtain Eq. 4, we reversed the starting and ending points of the first integral in Eq. 3, which reverses its sign. Using Eqs. 1 and 4 we obtain

$$\left(\frac{1}{2}mv^2\right)_i - \int_{\mathbf{r}_0}^{\mathbf{r}_i} \mathbf{F} \cdot d\mathbf{r} = \left(\frac{1}{2}mv^2\right)_f - \int_{\mathbf{r}_0}^{\mathbf{r}_f} \mathbf{F} \cdot d\mathbf{r} \ . \tag{5}$$

The fact that Eq. 5 holds for any choice of final position $\mathbf{r}_f$ means that the the quantity on each side of the equality has a fixed value for any choice of position $\mathbf{r}$ along the object's trajectory, or equivalently when measured at any time. We call that constant the *total energy* E, where

$$E \equiv \frac{1}{2}mv^2 - \int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{F} \cdot d\mathbf{r} \; . \tag{6}$$

We have already designated the first term on the right hand side of Eq. 6 as the kinetic energy; the second term defines the *potential energy* U:

$$U(\mathbf{r}) \equiv - \int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{F} \cdot d\mathbf{r} \; , \tag{7}$$

which is a function only of $\mathbf{r}$ (and the choice of the reference position $\mathbf{r}_0$), since we have assumed that $\mathbf{F}$ is a conservative force. Thus, we may rewrite Eq. 6 as

$$E = K + U \; , \tag{8}$$

and Eq. 5 as

$$E_{Initial} = E_{Final} \; , \tag{9}$$

which expresses the principle of the conservation of energy for conservative forces. In the next two sections, we find explicit expressions for the potential energy of a linear spring and for uniform gravity.

2.2 Conservation of energy and linear springs

Let's consider an ideal spring that obeys the force law $F = -ky$, where y is the distance the spring is stretched along its length. According to Eq. 7, the spring's potential energy is given by:

$$U_s(y) = - \int_{y_0}^{y} F(y')\, dy' = - \int_{y_0}^{y} -ky'\, dy' \tag{10}$$

$$= \frac{1}{2}k\left(y^2 - y_0^2\right) \; . \tag{11}$$

If we take the reference potential energy to be zero at $y_0 = 0$, then $U(y) = \frac{1}{2}ky^2$. If we keep one end of the spring fixed, stretch it horizontally, and measure the position y of the other end, $y_0 = 0$ is usually taken to be the position of that end when the spring is unstretched. It is also natural to define the potential energy as zero for the unstretched spring in this case. On the other hand, if we hang a spring from a support and attach a weight to the end, then depending on the problem to be solved, we might define $y_0 = 0$ for the spring position when no weight is attached or when the mass is hanging on it and stationary. Either works as long as the position and potential energy are defined consistently.

2.3 Conservation of energy in a uniform gravitational field

Consider a mass m in a uniform gravitational field, whose force is given by $F = -mg$, where the upward direction is taken to be positive. According to Eq. 7, the potential energy of the m in the gravitational field is given by:

$$U_g(y) = -\int_{y_0}^{y} F(y')\,dy' = -\int_{y_0}^{y} -mg\,dy' \tag{12}$$

$$= mg\,(y - y_0)\ . \tag{13}$$

If we take the reference potential energy to be zero at $y_0 = 0$, then $U(y) = mgy$. Note that the physical height to label y_0 is completely arbitrary, but once chosen it must be used consistently in all calculations.

The gravitational potential energy depends only on the vertical distance y the mass moves and not on the horizontal distance. Why?

2.4 Simple harmonic motion of a mass on a spring in a gravitational field

Consider a weight of mass m attached to the end of a spring hanging vertically down from a support. If the spring is stretched (for example, by pulling down on the weight) and then let go, the weight performs simple harmonic motion with an angular frequency given by $\omega = \sqrt{k/m}$, where k is the spring constant.

Define the coordinate y as the position of the bottom end of the spring and set it to be $y_0 = 0$ when unstretched, as it is when no mass is attached. Take the downward direction to be negative. In this case the total energy of the mass plus spring system can be written as

$$E = \frac{1}{2}mv^2 + \frac{1}{2}ky^2 + mgh\ . \tag{14}$$

Here h is the height of the center-of-mass of the weight, and is defined to be $h_0 = 0$ when the weight is in hanging position on the spring without any motion. Note that when the mass is hung on the spring, the spring stretches so that there is non-zero potential energy in the spring.

Alternatively, one could define $y_0 = 0$ as the spring position when the mass is hanging from it but not moving, in which case the spring potential energy is zero when the mass is not moving. Either definition works, as long as you are consistent in all calculations.

As energy is conserved, the total E in Eq. 14 should be the same at any point in the motion, assuming no friction.

3 Equipment used

In this laboratory you will investigate the conservation of energy for three different cases: free-fall in a uniform gravitational field, a pendulum, and a linear spring.

Miscellaneous items used Rubber tube (diameter $\approx$ 2.5 cm), paper tube (diameter $\approx$ 2.5 cm), $\sim$ 30-cm-long string, brass spring, hook masses, 500-g hook mass with index card taped to the bottom.

Mounting hardware Small bench clamp, 2 double clamps, 90-cm rod, 40-cm rod.

Motion sensor The motion sensor measures position as a function of time, and uses neighboring points to calculate velocity and acceleration.

Photogate sensor The U-shaped photogate sensor measures the time that a solid object first breaks a light beam and the time when it no longer breaks the beam. When you tell the software how thick the objcet is, it can calculate the speed with which the object passed by the sensor.

Other measuring devices Vernier calipers, 1-meter stick, 2-meter stick.

Software Capstone will be used for collecting the data from all sensors. Refer back to the instructions for "Motion 1 & 2 Experiment" if you need a reminder on how to use it.

4 Experimental procedures

4.1 Free-fall of a mass

When an object falls in a uniform gravitational field, its total energy should be conserved, provided all non-conservative forces are small compared to gravity. In this experiment you will investigate the free fall of a small rubber tube and a paper tube of roughly the same size.

A tube of mass m is held horizontally at a distance h above the level beam of a photogate sensor. The tube is dropped so that it falls without rotating and its velocity is measured as it passes through the photogate. The total energy just before it is dropped is compared to the total energy when the tube passes through the photogate.

4.1.1 Procedures

Measure the diameter of the rubber tube with the calipers, checking for uniformity. If the diameter of the tube is not uniform, discuss this in your error analysis. Program Capstone for **One Photogate (single flag)**. In the Hardware setup window click on the gear icon in the lower right corner. There, change the **Flag Width** to the diameter of the tube. Click ok. Click and drag **Digits** display to the center of the white screen. Click on **Select Measurments** and set it to speed. This will allow the software to calculate velocity from the photogate sensor measurements. At the top left of the digits window, increase or decrease precision until you have 3 decimal digits.

Attach the photogate to a supporting rod and rotate it so that its plane is horizontal and it is opposite the base plate that holds the supporting rod in place. Adjust the height of the photogate beam to be about 25 cm above the table. You will need to drop weights through the sensor making sure that they pass through the beam without hitting it. Can you come up with a way of making this easier? One possibility is to use the vertical rod that supports the spring (in the following experiment) to help you do the vertical allignment.

Drop the rubber tube taking care to make sure it doesn't rotate as it falls and remains horizontal as it passes through the photogate beam. Click Record, and drop the tube onto bubble wrap from a height of $h = 10$ cm above the beam. To what part of the tube should you measure the height? Record the velocity through the photogate. You will want to take some practice runs first. Repeat this measurement 20 times. Then do this 20 times for $h =$ 20 and 30 cm.

Do a similar experiment with the paper tube, dropping it a few times from heights of 10, 30, and 50 cm.

4.1.2 Analysis and questions to answer

1. Explain which part of the rubber tube you used to measure its height and why you chose this.
2. Define the quantity $\Delta E = E_{Final} - E_{initial}$. Write a python program that calculates the following for each of your sets of 20 measurements (rubber tube dropped from 20 cm and 30 cm, and paper tube dropped from 10, 30, and 50 cm).
   (a) ΔE for each measurement in the set.
   (b) The uncertainty in ΔE for each measurement in the set, obtained by propagating the uncertainties in all the quantities you measured.
   (c) The average value of ΔE.
   (d) The standard deviation of the measured set of events (this has 1/N in it).
   (e) The estimate of the standard deviation of distribution which describes the experiment (this has 1/(N-1) in it).
   (f) The standard deviation of the the average value of ΔE.
   (g) The fractional uncertainty in the standard deviation of the distribution (that is, the standard deviation of the standard deviation), which is given by $1/\sqrt{2(N-1)}$.
3. Are the standard deviations of the various measurements of ΔE consistent with your previous estimate of the uncertainty in ΔE?
4. Are the average value of ΔE and its uncertainty for each of the measurements consistent with what is expected from conservation of energy? Is there any friction in any of these experiments? If so, how would it affect your data?
5. Was the energy of the paper tube conserved when you dropped it? If not, what happened to the energy?
6. For each data set, how many measured events fall outside the range mean plus or minus one standard deviation?

4.2 Pendulum

In this experiment, the motion of the mass is not solely in the same direction as the conservative force, thus illustrating some new features of energy conservation.

4.2.1 Description

A 100 g mass is hung from a 30 cm string and used as a pendulum. The weight is pulled to one side and let go from various heights. At the lowest point the mass passes through the beam of a photogate sensor and its velocity is measured. The total energy at the beginning of the measurement is compared to the energy at the bottom of the pendulum's trajectory.

4.2.2 Procedures

Measure the diameter of the 100-g weight with calipers. Follow the procedures in the previous section to measure the velocity of the mass at the lowest point of its motion. Restart Capstone and program it for the photgate (single flag). As in the previous section adjust the **Flag Width** to the diameter of the 100-g weight. Then setup a digits display to measure velocity. Adjust the photogate on the lab bench so that the legs point up. The photogate beam should be directly below the suspension point of the pendulum. Adjust the height of the suspension point so that the full width of the weight swings through the center of the photogate beam, make sure to avoid the grooved portions of the weight passing through the beam (why is this important?). How high should the sensor be with respect to the weight? Pull the weight to one side so that it is 10 cm above the level of the beam. Let go and record the velocity of the weight at its lowest point. Repeat for $h = 8$, 6, and 4 cm. Take care to describe how you measure the change in height of the mass so that you are sure that you are measuring the change in height of the center of the mass.

4.2.3 Analysis and questions

1. Which part of the weight did you chose to put at sensor level? Why? Are your results sensitive to this choice?

2. How well is energy conserved? Is any deviation from perfect conservation within your experimental uncertainty? Does air resistance appear to play a significant role in this experiment or does it appear to be negligible?

3. Note that the mass has a force exerted on it by the string, but this force does not contribute to the potential energy. Why?

4.3 Gravity and a spring

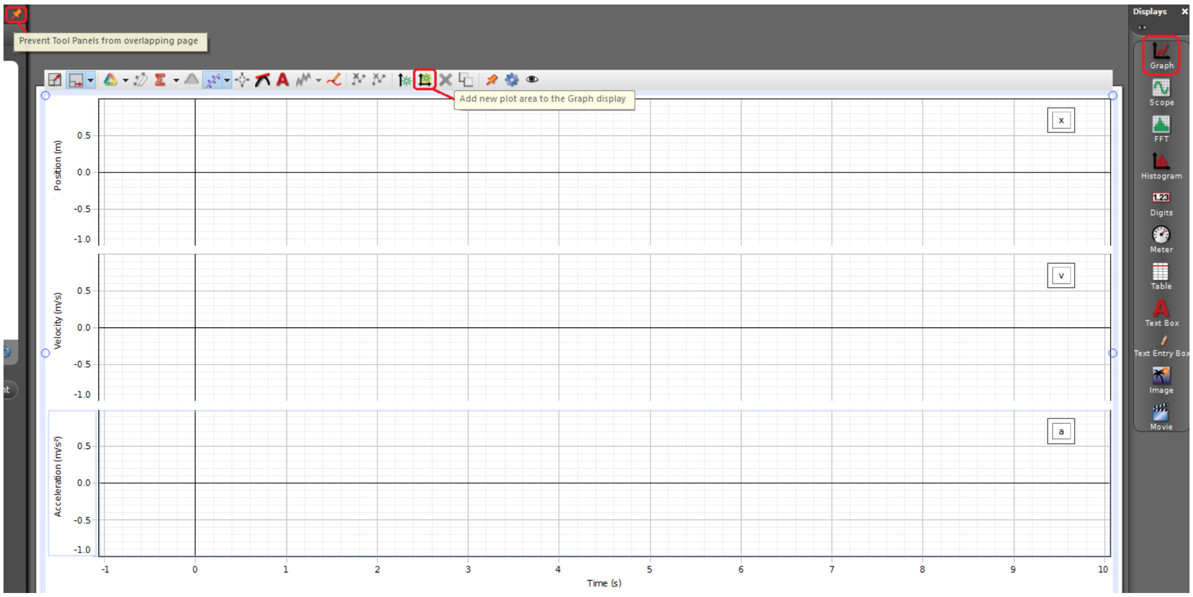
In this experiment you investigate conservation of energy for a case in which there are two conservative forces, that of a spring and of gravity, acting simultaneously.

4.3.1 Procedures

A 0.5 kg mass m is suspended from a vertical spring. The mass is set into vertical oscillatory motion and its position is measured as a function of time by a motion sensor. The velocity and acceleration are calculated by Capstone. The total energy of the mass is compared at different points in the motion. Here the potential energy of the spring-mass system is due to both gravitational potential energy and the potential energy of the spring.

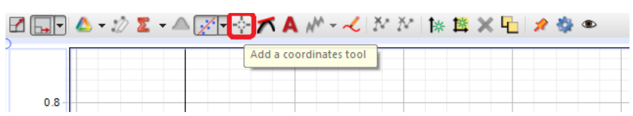
Hang the spring over the edge of the bench. Tape the top of the spring to the rod from which it is suspended so that the spring cannot slip off the rod inadvertently. Adjust the height of the spring so that when attached, the bottom of the 0.5 kg mass is 60–70 cm from the floor. Measure the force constant k for the spring. Place the motion sensor on the floor directly beneath the the mass with the grill facing up and horizontal. There is a switch on the motion sensor which sets the width of the acoustic beam of the sensor. You will probably find that the narrower width works the best, but feel free to experiment. The motion sensor should be plugged into the interface so that position increases as the mass goes higher.

Restart Capstone and program Capstone for the motion sensor, using the default values. Drag and drop the graph icon from the displays column onto the white screen. Add two additional plots by using "Add a new plot area to the graph display." Set up position, velocity, and acceleration on the graph display as the vertical axes. You will probably find the default sampling rate of 20 Hz to be satisfactory, but feel free to experiment. Look at the following illustration for guidance.



Set the mass into motion by pulling it down a reasonable distance and letting go. Recall that the motion sensor does not record distances less than 0.15 m from its face. Do not set the mass in motion by lifting it up, since if you lift it too far it will crash into the motion sensor. Wait until the motion settles into a regular pattern to do the following.

Click RECORD, and click STOP after 3 or 4 cycles of the motion. Using the **Add a Coordinates Tool** on the position graph.



Determine the distance the mass traveled from a chosen highest position (H) to the following lowest position (L) on the position graph. This distance will be $2A$, where A is taken to be a positive number. Determine the total energy E for the following 4 positions of the mass during the motion: the highest and lowest positions of the mass utilized in finding the amplitude A, and the two positions of maximum velocity following these highest and lowest positions of the mass. Recall that the coordinate y gives the position of the end of the spring, not the position of the mass. The highest position of the mass occurs when $y = y_M + A$ and the lowest position of the mass when $y = y_M - A$, where $y = y_M$ is the midpoint position. Maximum velocity occurs when $y = y_M$, at which time the mass is at a height h_0.

1. When the mass is at the highest chosen position and the kinetic energy is zero, the energy E_H will be
$$E_H = \frac{1}{2}k(y_M + A)^2 + mg(h_0 + A). \tag{15}$$

2. When the mass has the maximum down velocity following the highest chosen position of the mass the energy E_{01} is given
$$E_{01} = \frac{1}{2}mv^2 + \frac{1}{2}ky_M^2 + mgh_0. \tag{16}$$

3. When the mass is at the lowest chosen point and the kinetic energy is zero, the energy is given by
$$E_L = \frac{1}{2}k(y_M - A)^2 + mg(h_0 - A). \tag{17}$$

4. When the mass has the maximum up velocity following the lowest chosen position of the mass the energy E_{02} is given by
$$E_{02} = \frac{1}{2}mv^2 + \frac{1}{2}ky_M^2 + mgh_0. \tag{18}$$

Print out the position, velocity and acceleration graphs.

4.3.2 Analysis and questions

1. How did you measure the force constant of the spring?

2. Observe the motion of the mass study and the graphs of position, velocity, and acceleration. Is the acceleration maximum or minimum when the velocity is zero? When the velocity is maximum is the acceleration maximum or minimum?

3. Observe the spring-mass system. Is there any kinetic energy that is not accounted for by $\frac{1}{2}mv^2$? If so, this would be worthwhile estimating quantitatively in your error analysis.

4. Compare the total energy at the 4 points described above (mass at highest and lowest positions, maximum down velocity and maximum up velocity). Are your measurements consistent with energy being conserved? What factors might introduce error into your measurements and calculations?

5. The analysis presented in this section assume no friction. How will friction affect your experimental results? Do you see evidence of friction in the curve of position *vs* time?

5 Finishing Up

Please return the bench to the condition in which you found it.