

# Spectrum Analysis: The FFT Display

Equipment: Capstone, voltage sensor, microphone, tuning fork, bass guitar

## 1 Introduction

It is often useful to represent a function by a series expansion, such as a Taylor series. There are other series representations of functions that are also very useful. A function can be expanded in a series of sines and cosines, which is called a Fourier series. This may be more familiar than you think. In music, a pure tone is a pressure that varies sinusoidally with time, and is characterized by a single frequency. Frequency is measured in cycles or oscillations per second, which, in honor of Heinrich Hertz, is called hertz (Hz). A young ear in good condition (not damaged by music that is too loud) can hear pure notes that range from about 15 Hz to 18 kHz. Music is composed of many pure tones being played at the same time. The resulting sound depends on the amplitude and phase of each pure tone present in the sound. The sinusoidal tones of the various frequencies present at any time are the “frequency spectrum” of the music. In this lab you will investigate the frequency spectrum of “white noise” and of a number of voltages that are periodic in time. Specifically, the voltages are the sine, square, triangle, and ramp (sawtooth) waves. To accomplish this the FFT display will be used. FFT stands for fast fourier transform. Without going into very much detail, the theory section will give you an idea of what the FFT does.

## 2 Theory

### 2.1 The Fourier Series

We will consider the Fourier series for a function  $f(t)$ , where the independent variable  $t$  stands for time. (The independent variable could be anything, such as  $x$  for distance.) Any reasonable function  $f(t)$ , even if it has discontinuities, can be expanded in a Fourier series over any time interval  $P$ . If  $f(t)$  is a periodic function of period  $P$ , the series will represent  $f(t)$  everywhere. The sines and cosines of the Fourier series have as arguments the fundamental frequency  $\nu = 1/P$  and integer multiples of  $\nu$ . The series may be written

$$f(t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} [A_n \cos(2\pi n\nu t) + B_n \sin(2\pi n\nu t)], \quad (1)$$

where the  $A_n$  and  $B_n$  are constants that depend on the form of  $f(t)$  and the  $n$  are positive integers. The “amplitude” of a Fourier component of frequency  $n\nu$  is given by  $\sqrt{A_n^2 + B_n^2}$ . Due to the definition of  $\nu$  as  $1/P$ , all the sinusoids have period  $P$ . (For sinusoids with arguments of  $n > 1$ , there will also be shorter periods.) The frequency  $\nu$  is called the fundamental (frequency), the frequency  $2\nu$  is called the 2nd harmonic,  $3\nu$  the 3rd harmonic, and so forth.

The constants  $A_n$  and  $B_n$  are obtained by using the “orthogonality” relations of the sinusoidal functions. These relations are, where  $m$  and  $n$  are integers,

$$\int_{t_0}^{t_0+P} \sin(2\pi m\nu t) \sin(2\pi n\nu t) dt = \begin{cases} 0 & m \neq n \\ \frac{P}{2} & m = n \neq 0 \\ 0 & m = n = 0 \end{cases}, \quad (2)$$

$$\int_{t_0}^{t_0+P} \cos(2\pi m\nu t) \cos(2\pi n\nu t) dt = \begin{cases} 0 & m \neq n \\ \frac{P}{2} & m = n \neq 0 \\ P & m = n = 0 \end{cases}, \quad \text{and} \quad (3)$$

$$\int_{t_0}^{t_0+P} \sin(2\pi m\nu t) \cos(2\pi n\nu t) dt = 0. \quad (4)$$

For example, to find the coefficient  $A_m$ , multiply both sides of Eq.(1) by  $\cos 2\pi m\nu t$  and integrate every term on both sides over the period  $P$ . Every term on the right hand side is zero except the term containing  $A_m$ . The results are

$$A_n = \frac{1}{P} \int_{t_0}^{t_0+P} f(t) \cos(2\pi n\nu t) dt, \quad (n = 0, 1, 2, 3, \dots), \quad (5)$$

$$B_n = \frac{1}{P} \int_{t_0}^{t_0+P} f(t) \sin(2\pi n\nu t) dt, \quad (n = 1, 2, 3, 4, \dots). \quad (6)$$

In practice, the period  $P$  is almost always taken between 0 and  $P$  or between  $-\frac{1}{2}P$  and  $+\frac{1}{2}P$ . The choice of origin, which can sometimes make a function even or odd, can simplify the calculations of  $A_n$  and  $B_n$ .

Summarizing this section, over a time interval  $P$ , a function  $f(t)$  can be represented by a sum of sinusoids of various frequencies, or a Fourier series. The lowest frequency present, the fundamental, is  $\nu = 1/P$ . Higher frequencies are integer multiples of the fundamental. A spectrum analyzer determines the amplitudes of the frequencies present.

### 3 Some Fourier Series

A single sine wave is its own fourier series. There is just one term. The Fourier series of the other wave forms involved in this lab are given below and are also shown in Fig. 1. It is assumed the waveforms have unit amplitude, frequency  $\nu$  and an average value of zero.

1. square wave:  $\sum_n \frac{4}{\pi n} \sin 2\pi n\nu t \quad (n=1, 3, 5, 7, \dots)$
2. triangular wave:  $-\sum_n \frac{4}{\pi^2 n^2} \cos 2\pi n\nu t \quad (n=1, 3, 5, 7, \dots)$
3. ramp wave:  $\sum_n \frac{2(-1)^{n+1}}{\pi n} \sin 2\pi n\nu t \quad (n=1, 2, 3, 4, 5, \dots)$

The Fourier series for the square and triangular waves contain the fundamental and odd harmonics. The series for the ramp wave contains all harmonics.

## 4 Noise

If one is looking for a signal, particularly a low level one, noise may be the limiting factor in detecting the signal. (Not enough amplification might be another limiting factor.) In listening to a radio, noise from the electrical activity of thunder storms, rotating electrical machinery, and for car radios, power lines, are common. This kind of noise can be called interference, the picking up of a signal you do not want. But there are other kinds of noise which are more fundamental. We discuss white noise.

### 4.1 White Noise

White noise is characterized by constant power per unit bandwidth. To explain what this means, we assume that it is possible to filter an electrical signal so that only frequencies between two values remain. That range of frequencies is called the bandwidth. If a signal is white noise, there will, for example, be the same power between 100 and 105 Hz, as there is between 10,000 and 10,005 Hz, where the band width is 5 Hz. White noise contains all frequencies. All the Fourier components of white noise are completely random with respect to each other. This is very different from the periodic functions just discussed which have only certain frequencies present, and these frequencies have a very definite phase with respect to each other. A spectrum analyzer will show that white noise contains all frequency components and that the discussed periodic functions only have certain discrete frequencies.

An example of a device that produces white noise is a resistor. The white noise from a resistor is often called Johnson noise. Johnson noise arises from the voltage fluctuations across the resistor due to the random thermal motions of the electrons in the resistor. The root mean square voltage fluctuations of a resistor  $v_{rms}$  is given by  $\sqrt{4kTRB}$ , where  $k$  is Boltzmann's constant,  $T$  is the absolute temperature,  $R$  is the resistance, and  $B$  is the bandwidth of the device measuring the voltage.

## 5 The Discrete Fourier Transform (DFT)

The DFT is also known as the digital Fourier transform. The DFT is arrived at by dividing the interval  $P$  into  $N$  equal parts. The integrals that give the Fourier coefficients  $A_n$  and  $B_n$  are then approximated by a sum of  $N$  terms. (The DFT uses a different normalization so these approximated Fourier coefficients are proportional to the DFT.) Roughly speaking, if as before  $n$  is the integer that represents the harmonic, the approximation is only good for  $n \leq \frac{1}{8}N$ . The larger the  $N$ , the better the approximation. The DFT does not have to approximate a function. It can represent  $N$  values of something, such as a voltage sampled  $N$  times at regular intervals. That is what the FFT (Fast Fourier Transform) display does. It samples the input voltage  $N$  times over a time interval  $P$  and computes the DFT. The amplitudes of the DFT represent the frequencies present in the voltage, or the frequency spectrum.

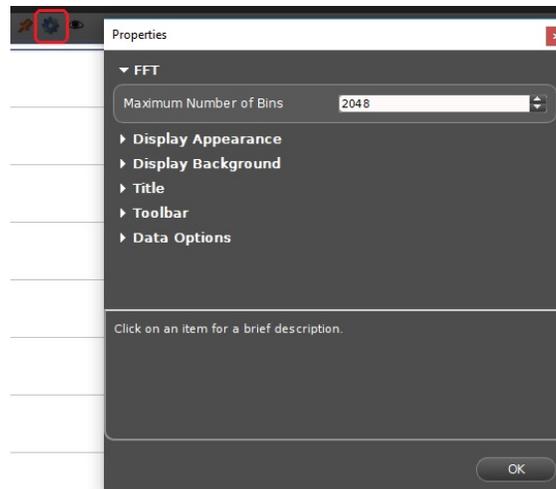
## 6 Fast Fourier Transform (FFT)

If  $N$  is the number of voltage samples or data points, the number of operations necessary to calculate the DFT is about  $N^2$ . This rapidly becomes prohibitive as  $N$  gets large. Usually  $N$  is taken as a power of 2, so that  $N=2^M$  (radix 2). The FFT is an algorithm that reduces the number of operations necessary to calculate the DFT. The savings as  $N$  becomes large are considerable.

## 7 The FFT Display

The FFT display samples a voltage  $N$  times the sampling rate. The DFT for the  $N$  samples are calculated using a FFT. The results are displayed in the window of the FFT display, and the entire process is continually repeated and the display continually updated. The frequencies present are the frequency spectrum, or if you wish, the frequency distribution. The horizontal axis of the FFT display window is frequency, and the vertical axis represents how much frequency amplitude is present in certain frequency bands.

The number of bins  $N$  can be chosen by clicking on gear icon on the FFT display.



The sampling rate is controlled at the bottom center of the display area. The sampling rate itself is not shown in the FFT display window. What is shown is the maximum frequency that can be measured which is half the sampling rate, and this can be determined by the scale at the bottom of the display area.

After collecting the samples, the display calculates the DFT. After the display does this, it presents the results in the display area. If the number of bins  $N$  is large, and the sampling rate or maximum frequency is low, it can take considerable time for the FFT display to collect the samples. Consequently, it can take some time for the display to be updated. For some of the parameters used, you can easily see the display remain the same for a while, and then suddenly change to a new picture. As an example, suppose  $N=1024$ , and the maximum frequency of the FFT display is 100 Hz. The sampling rate  $SR$  is 200 Hz. The time to take  $N$  bins will be  $N/SR=1024/200=5.12$  s. To this would have to be added the calculation time.

## 8 Using The FFT Display

For some of the sections, you will be looking at the spectrum of the signal generator when the generator is off. Under these circumstances, the observed signal is mostly white noise. The voltage level is of the order of mV.

When you open the FFT set-up window, the gain of the FFT display is automatically adjusted so that the fundamental of the voltage being observed has a full scale value on the display screen. This write-up will guide you in choosing a maximum value, but change it if necessary. Below are two buttons that you will use often.



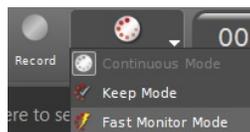
The first button is useful if you have an applied frequency that is in the high or off scale end of the FFT display. The second button normalizes the display.

### 8.1 Maximum Frequency and Sampling Rate

The relationship between the maximum frequency displayed by the FFT display and the sampling rate will be investigated. Open the signal generator window by clicking on the signal generator icon and select 850 Output 1. Leave the signal generator *off*. Go to the Hardware Setup window and on the 850 interface display and click on Output 1 and select **Output Voltage-Current Sensor**. Open a FFT display in the right experiment set-up window. In the vertical axis select **Output Voltage-Current Sensor**. Scroll the x-axis of the FFT display and set the maximum frequency in the FFT window to around 50 Hz. **What is the sampling rate?** Repeat for a maximum frequencies of around 100 and 250 Hz. **What is the relationship between the maximum frequency and the sampling rate?**

### 8.2 Number of Samples or Points, N

The relationship between the number of data points N and the number of frequency bands will be explored. **Leave the signal generator off.** You will be looking at the spectrum of white noise. Open a new FFT set-up window and set the number of data bins N at 32. Scroll the maximum frequency to around 100 Hz. Go to the bottom of Capstone and switch from **Continuous mode** to **fast Monitor mode**.



Click the MON button and then click on **Normalize data** button on top and observe how the display updates itself. Click stop, and count the number of peaks between 0 and 100 Hz. Repeat for an N of 64 and 128. **How is the number of peaks or bands related to N?**

For N=32, count the number of peaks for maximum frequencies of around 250 and 500 Hz. **Does the number of bands depend on the maximum frequency?**

### 8.3 White Noise

Leave the signal generator off. In the FFT set-up window, make N=1024 and *leave it at this value for the rest of the experiment.* You can use that value for the rest of this lab. **Also, you need to use the normalize button in the FFT display.** Look at the spectrum of white noise for a number of maximum frequencies. **Is the spectrum more or less what you would expect? When the display updates, is it the same?**

**How big are the amplitudes for the noise?** Right click on the vertical axis, go to axis properties and set the axis with a maximum value of 0.5 mV. Give a rough value of the fourier amplitudes.

## 8.4 Frequency Spectrum of Periodic functions

In the set-up window for the FFT display, go to the vertical axis and set it with a maximum value of 1 V. In the signal generator window, click auto, set the frequency amplitude to 1 V, and the frequency to 100 Hz. First look at the frequency spectrum of the sine wave, choosing a number of appropriate maximum frequencies. **Do you see any harmonics?** Note the amplitude of the fundamental.

Look at the frequency spectrum of the triangle and ramp waves. Compare the amplitudes, relative to that of the fundamental, with the values given in the theory section. The comparison will not be exact. There are a number of reasons for this. Among them, the signal generator does not put out exact wave forms, and the DFT only approximates the fourier coefficients. Discuss the differences between the theory and what is given by the spectrum analyzer. **Are there harmonics missing? How do the ratios of the fundamental amplitudes compare?**

**Now look at the frequency spectrum of the square wave. Is the maximum value of 1 V on the vertical axis of the FFT display large enough?** Discuss the spectrum of the square wave and compare to the theory.

## 8.5 Interference or Pick-Up

There are all kinds of low level electromagnetic waves from a multitude of sources in the lab room. Any wire will act as an antenna. Let's take a look at this. For the FFT display, set the maximum frequency at 500 Hz and set the vertical axis to a maximum value of 5 mV. Hook up a voltage sensor to channel A of the interface and program the interface. Just leave the two leads of the voltage sensor unconnected on the bench. Click the input menu button on the FFT display and input the voltage sensor from channel A. Look at the spectrum. You will probably see some odd frequencies you cannot account for. Plug the two ends of the voltage sensor leads together to short them. **Does the pick-up level go down?** With the leads shorted, does the noise level depend on whether you make a big loop with the leads or if you keep the two wires close together. (You almost always can reduce pick-up by twisting your test leads together. Using coaxial cable will reduce the pick-up even more.) **What happens to the noise if you remove the voltage sensor plug from the interface?**

If you are doing an experiment where the signals are low and noise is a real problem, knowing the spectrum of the noise can be very useful, as it can guide you in how to reduce it or get around it. Twisting leads together or using coax are standard ways of reducing pick-up. Unplugging things is one way of finding out what is responsible for the unwanted signal you might be observing.

## 8.6 FFT of Pure Tones

Attach a microphone to your voltage sensor and adjust your readout in the Capstone accordingly. Now, proceed as above with the following measurements: (a) record the sound of the tuning fork on your bench and save as a csv file, (b) record the respective tones of all 4 strings of the base guitar and save as separate csv files, (c) record the 4 tones of 4 respective strings of another musical instrument in the lab, e.g. violin or ukulele and save as separate csv files, (d) record 2 combinations of 2 to 3 different strings and save as separate csv files. Make sure you choose appropriate sampling frequency for (a-c), which allows you to detect the signal. The signal should be clearly visible and

have a distinct shape in the frequency space. Determine frequencies of all pure tones in (a-c). Can you find the frequencies of the pure tones in (d)?

## 8.7 FFT of Human Voice

Next, we will analyze the frequency spectrum of human voice: Record your own voice while (e) saying sentence and (f) while reading a poem or children's rhyme. Can you find the dominant frequencies of your voice in (e) as well as the frequencies of the rhyme in (f)? Save the recordings of (e) and (f) as csv files.

## 8.8 FFT in Python

So far, we have used for all questions the FFT Display to access the frequency space and the results of the FFT. It is very useful to learn to perform the FFT outside of Capstone and understand the form of the data after the FFT is performed. To do so, you can follow a simple procedure described in the *Python Manual* by David Pine (starting on p. 173). Once you adapt the algorithm mentioned in the manual, you can run it on your data that you saved as csv files in Sections 8.6 (a-d) and 8.7 (e-f). Determine the frequencies once again using your Python code. Do they agree with the ones found using the FFT Display? Insert the printout of your Python code as a part of your lab report.

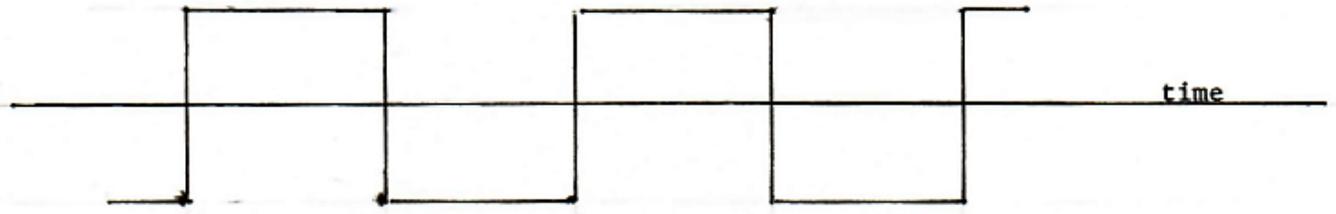
## 9 Questions

1. **Where is the origin chosen for the fourier series of the square, triangle, and ramp functions given in the theory section? (Hint: Are the given series representing even or odd functions?)**
2. For the fourier series of the square, triangle, and ramp functions, two of these have coefficients that fall off as  $\frac{1}{n}$ , and one has coefficients that fall off as  $\frac{1}{n^2}$ . **Could you guess as to why?**
3. With respect to the fourier series, show that  $A_n \cos(2\pi nvt) + B_n \sin(2\pi nvt)$  can be written as  $C_n \sin(2\pi nvt + \phi)$ . Find  $C_n$  and  $\phi$ . **Do you think the definition of the amplitude of a fourier coefficient given in the theory section is appropriate?**
4. **If you put a periodic function into the spectrum analyzer, can you reconstruct the function from the frequency spectrum?** Discuss.
5. Does it seem reasonable that Johnson noise increases as the temperature increases. Discuss.

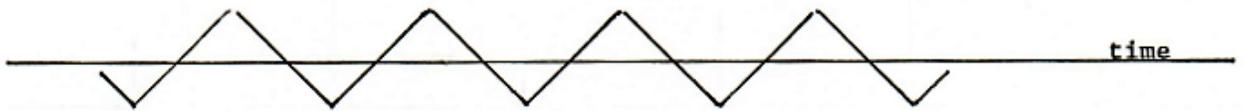
## 10 Finishing Up

Please leave the bench as you found it. Thank you.

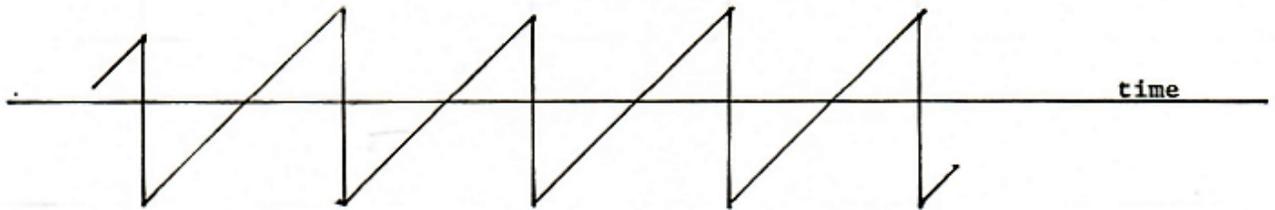
### Spectrum Analysis: The FFT Display



Square Wave



Triangular Wave



Ramp Wave

Figure 1: