

Oscillations Of A String

Equipment: Capstone, mechanical wave driver, meter stick, analog scale, elastic cord, pulley on rod, 2 rods 30 cm or more, 1 large table clamp, 1 small table clamp, dual rod clamp, set of slotted weights, weight hanger, leads [180 cm (2)], cardboard, stroboscope.

1 Introduction

A taut string can be made to vibrate. If the string is plucked and then left alone, the oscillations of the string are “free oscillations.” If the string is subjected to a time dependent external driving force, the oscillations are “forced.” The coupling between the string and the driving force can vary continuously from weak to strong. In this experiment, the oscillations of a string will be studied when subjected to a weakly-coupled driving force.

If a system is displaced from its equilibrium configuration and let go, it will vibrate. The system will vibrate in one or more of its normal modes (eigenmodes, or modes for short). In a given mode, all parts of the system vibrate sinusoidally at the same frequency and pass through their equilibrium positions at the same time, i.e., have the same phase. Except for degeneracy, each mode has a distinct frequency. There are as many normal modes as there are degrees of freedom of the system. A continuous system has an infinite number of modes. Any oscillations of a system can be described by a linear combination of its normal modes.

If a system is weakly coupled to a sinusoidal external driving force it is found that the system response is much larger at the normal mode frequencies than at other frequencies. This is called a “resonance”.

2 Theory

We consider a string fixed at both ends and assumed to vibrate in the $x - y$ plane. See Fig. 1. The length along the string is designated by x , with the string fixed at $x = 0$ and $x = L$. The transverse displacement of the string is called $y = y(x, t)$. The string is assumed to have a uniform constant tension T and a uniform mass per unit length of ρ . If Newton’s 2nd law in the y direction is applied to a length dx of the string, and if no damping is assumed, the equation describing the string is found to be the wave equation,

$$\frac{\partial^2 y}{\partial x^2} - \frac{\rho}{T} \frac{\partial^2 y}{\partial t^2} = 0. \quad (1)$$

The most general solution to this equation is $g(x \pm vt)$, where g is any function and $v = \sqrt{T/\rho}$. These solutions are waves traveling in the positive or negative x direction with speed v . The boundary conditions of our problem limit the possible solutions. We look for sinusoidal normal mode solutions and guess a solution of the form $y = f(x) \cos \omega t$, where ω is the angular frequency

and t is the time. We shortly find that only certain discrete values of ω satisfy the boundary conditions. Substituting this assumed solution into Eq.(1),

$$\frac{d^2f}{dx^2} + \frac{\rho}{T}\omega^2 f = 0. \quad (2)$$

The solution to this equation is a sum of sines and cosines, but only the sines will satisfy the boundary condition $y(0, t) = 0$. Assume $f(x)$ to be of the form $A \sin \frac{2\pi}{\lambda}x$, where A and λ are constants. The boundary condition $y(L, t) = 0$ can only be satisfied if

$$\lambda_n = \frac{2L}{n}, \quad n = 1, 2, 3, \dots, \quad (3)$$

where the subscript on λ indicates which of the allowed values it has. A given n refers to a normal mode, and we also subscript A as A_n . If $A_n \sin \frac{2\pi}{\lambda_n}x$ is substituted into Eq.(2) we can find the allowed or normal mode angular frequencies ω_n . Using frequency ν in Hz rather than ω in radians/s, from $\omega = 2\pi\nu$ the normal mode frequencies are

$$\nu_n = \frac{n}{2L} \sqrt{\frac{T}{\rho}}. \quad (4)$$

The normal mode vibrations of a string are often called standing waves. Standing waves are produced on a string if two sinusoidal waves of the same frequency and amplitude are traveling on the string in opposite directions. There are stationary points on the string separated by $\lambda/2$ where the string does not vibrate, or $y = 0$. These points are called nodes. The 2 fixed end points of a string are nodes. For the lowest frequency or 1st mode, there are no other nodes. Mode 2 has one extra node, mode 3 has 2 extra nodes, and so forth. The first 4 modes are shown in Fig. 2.

The solid line shows the string at an instant in time when the amplitude is maximum and all points of the string are instantaneously at rest. A quarter of a period later the string is flat but all points of the string except the nodes are moving. A half period later the string is shown by the dotted line. The envelope of motion, what you actually see in the lab consists of both the dotted and solid lines. The anti-nodes are those points along the string which have maximum amplitude. The anti-nodes are situated midway between the nodes.

For a given sinusoidal wave on the string, the usual wave relationship $\lambda\nu = v$ applies. If $g(x \pm vt)$ is substituted into Eq.(2), it is found that the velocity of waves on the string is $v = \sqrt{T/\rho}$. For a given T and ρ , the speeds of the waves are a constant independent of the frequency or wavelength. The waves are said to be dispersionless.

In the experiment, normal modes will be excited by pushing the side of an oscillating rod lightly against the string. The rod will be applied near one end of the string, and the coupling is due to friction between the rod and string. The rod is driven with a sine-wave pattern. Due to damping of the string, there will also be a response of the string for frequencies away from the normal mode frequencies, but the response will fall off in a typical resonance curve fashion as the applied frequency moves further from the resonant frequency. The size of the amplitude of the oscillation and phase of the motion (relative to that of the driving rod) are shown in Fig. 4. The amplitude is given by

$$1/\sqrt{(\omega^2 - \omega_0^2)^2 + \gamma^2\omega^2}, \quad (5)$$

where γ is the *damping* of the string. (The phase difference (lag) will be hard to measure in this lab, since you can't accurately measure the string's phase.) The damping is related to how fast the amplitude decreases vs. time, as shown in Fig. 5. The envelope (dashed lines) around the motion

are given by $A_0 e^{-\gamma t/2}$. Sometimes people refer to the "Q" of a system, which is the ratio of the stored energy in the system / work done by the driving force during each radian. $Q = \omega_0/\gamma$.

3 Experiments

3.1 Equipment

The oscillations of the string will be driven, but weakly enough coupled (the string is not fixed to the oscillating rod) so that the normal modes will not be substantially affected. The string used is actually an elastic cord which is quite visible and produces good patterns. The Mechanical Wave Driver (vibrator) creates the oscillations, and is driven by the interface.

3.2 Preliminary Measurements and Set-Up

Fig. 3 is a sketch of the apparatus. The measurements you take are necessary to calculate the normal mode frequencies from Eq.(4).

- At one end of the lab bench, use the large table clamp to secure the pulley vertically.
- Using the dual rod clamp and small table clamp, attach the two rods perpendicularly, and mount the vertical end to the table.
- Measure L , the distance between the center of the fixed rod and the top of the pulley.
- Weigh the cord, and measure its unstretched length. Calculate the unstretched mass per unit length ρ_0 of the cord.
- Wind one end of the cord several times around the fixed horizontal rod and secure it with a few half hitches. The other end of the cord goes over the pulley.
- With no tension on the cord, measure the vertical length of the cord from the top of the pulley to the small loop in the cord. Hang a mass of $M = 400$ g from the cord (this includes the mass of the weight hanger), and measure how much the cord stretches. Using these measurements and L , you can calculate the mass density ρ of the cord when it has been stretched for that M . Remove the masses.
- Place the vibrator on the bench. The rod of the vibrator is vertical and oscillates up and down. The height of the horizontal rod should be adjusted so that the cord is about half-way between the top of the vibrator body and the shoulder of the rod.
- Adjust the position of the vibrator so that the center of the vibrator rod is 5 cm from the center of the fixed rod and so that the vibrator rod just touches the cord.
- Move the vibrator so that its rod moves the cord 0.5 cm in the transverse direction.
- With the two leads, connect the vibrator to the Capstone interface Output 1.
- Click on the Signal Generator icon in the Tools column of Capstone. You will be using sine waves with frequencies between 5 and 100 Hz. The output voltage of the interface should be about 3 or 4 V for these experiments.

Be sure the lock on the vibrator is off when you supply voltage.

3.3 Dependence of ν_1 on T

Put a mass of 100 g on the cord (reminder: measure the stretch of the cord for this M and calculate ρ). Supply a 4 V sine wave to the vibrator and vary the frequency until mode 1 is at a maximum. Record this frequency. Repeat for masses of 200 g and 400 g on the cord. Calculate the frequencies predicted from Eq.(4). Make a table of your experimental and theoretical results. **How well to they agree?**

3.4 Dependence of ν_n on Mode

Put 200 g on the end of the cord. Find the experimental normal mode frequencies for modes 1-4. **Are these frequencies related in the way you expect from Eq.(4)?**

3.5 Resonance Shape

With 200 g on the end of the cord, investigate the response of the system for frequencies near ν_1 and near ν_2 . Record the maximum amplitudes as a function of frequency. **Do you see resonance curves like Fig. 4 (left)?** Measure γ from your resonance curve and Eq. 5. Recall that γ is approximately the full-width at half-maximum (FWHM) of the curve.

3.6 Damping

Set the string vibrating in the 1st mode. Turn the vibrator off and observe how long it takes for oscillations to die away. Measure the amplitude as a function of time (relative to when you turn off the oscillator), as precisely as you can. **Does it look like Fig. 5? Extract the γ damping factor for your string by fitting to your exponential decay curve measurements. Does it agree with the γ you measured from the resonance curve?**

3.7 Transverse Mode

The string is driven in the vertical direction and would be expected to vibrate in a vertical plane. You may find that at and near resonance the string has horizontal as well as vertical oscillations. Look for this by looking at the string not only from the side but from the top. When the string vibrates this way each element of the string is moving in a circle about the equilibrium position of the string. Use the stroboscope to examine the motion. You should take the maximum amplitude of this motion as an indication of resonance. The frequency is close to or the same as the frequency of the mode in which the string vibrate in a vertical plane. **What is causing this transverse mode to be excited?**

4 Your own experiment

Try something new with this setup! You can extend the apparatus if needed, or use it in different ways. Describe your experiment in detail, show your measurements, and compare to theoretical expectations if possible.

5 Conclusions

Discuss your experiments and the results. **Question: What is another system in the world with damped, driven, oscillations, and how do its behaviors relate to the effects mea-**

sured in this experiment?



