Coupled Pendulum

Equipment: 2 Rotary Motion sensors, 3 rods, 2 bench clamps, 2 rod clamps, 2 mini-rotational rods, 2 brass cylinders, size 18 rubber band, and a ruler.

1 Introduction

You are all familiar with the simple harmonic motion of a single pendulum. In this lab, you will experiment with the motion of two simple pendulums coupled by a rubber band, which allows the two pendulums to interact. While the general motion of this system might seem complicated, any motion of this system can be described by the sum of two normal modes.

If the two pendulums are identical, and the initial conditions are chosen appropriately, the available energy passes back and forth between the two pendulums. This feature will be examined and is known as "beats." The idea of normal modes in linearized systems is very useful and occurs in many diverse areas of physics, such as quantum mechanics.

2 Theory

We obtain the equations of motion for the system of two pendulums, designated a and b, which are coupled by an elastic force, with the following assumptions and approximations:

- The two pendulums are identical and, when uncoupled, have the same natural frequency.
- Each pendulum oscillates in a plane and has one degree of freedom.
- The angular displacements of the pendulums from vertical are given by θ_a and θ_b. If the pendulums are displaced in the same direction, both these angles have the same sign.
- The pendulums are simple. The pendulum rods are massless and are of length ℓ. The masses, denoted by M, are points.
- The deflection angles are small, and the equations of motion are linearized: $\sin\theta \cong \theta$.
- Let K be a constant. The force on pendulum a in the direction of motion due to the coupling of the rubber band is $-K\ell(\theta_a - \theta_b)$, and the force on pendulum b due to the coupling is $-K\ell(\theta_b - \theta_a)$. This assumes that the rubber band is operating in its linear regime, where it is stretched just a small amount compared to its maximal stretch.
- There is no friction or energy loss of the system.

The equations of motion become

$$\frac{d^2}{dt^2}\theta_a + \frac{g}{\ell}\theta_a + \frac{K}{M}(\theta_a - \theta_b) = 0, \qquad and \tag{1}$$

$$\frac{d^2}{dt^2}\theta_b + \frac{g}{\ell}\theta_b + \frac{K}{M}(\theta_b - \theta_a) = 0. \tag{2}$$

We assume that it is possible to find values of ω such that

$$\theta_a = A\cos(\omega t + \phi) \qquad and \tag{3}$$

$$\theta_b = B\cos(\omega t + \phi) \tag{4}$$

are solutions to Eqs. 1-2. The angular frequency of the motion is ω. A, B, and ϕ are constants

which can be chosen to satisfy initial conditions. Substituting Eqs. 3-4 into Eqs. 1-2 yields the following algebraic equations.

$$\left(\frac{g}{\ell} + \frac{K}{M} - \omega^2\right)A - \frac{K}{M}B = 0, \quad and \tag{5}$$

$$-\frac{K}{M}A + \left(\frac{g}{\ell} + \frac{K}{M} - \omega^2\right)B = 0. \tag{6}$$

These two homogeneous equations have solutions only for the following two values of ω:

$$\omega_1^2 = \frac{g}{\ell} \quad and \quad \omega_2^2 = \frac{g}{\ell} + \frac{2K}{M}. \tag{7}$$

For ω_1, or mode 1, either Eq. 5 or Eq. 6 yields $A_1 = B_1$. The subscripts on A and B indicate

that these are the amplitudes associated with mode 1. This is a symmetric mode where the pendulums have the same amplitude and the same phase. For ω_2, or mode 2, either Eq. 5 or Eq. 6 yields $A_2 = -B_2$. The subscripts on A and B indicate that these are the amplitudes associated with mode 2. This is an antisymmetric mode where the pendulums have the same amplitude but opposite phases.

The motions for mode 1 are given by

$$\theta_{a1} = A_1\cos(\omega_1 t + \phi_1) \quad and \quad \theta_{b1} = A_1\cos(\omega_1 t + \phi_1). \tag{8}$$

The motions for mode 2 are given by

$$\theta_{a2} = A_2 \cos(\omega_2 t + \phi_2) \quad and \quad \theta_{b2} = -A_2 \cos(\omega_2 t + \phi_2). \tag{9}$$

Any motion of the linearized system is given by

$$\theta_a = \theta_{a1} + \theta_{a2} = A_1 \cos(\omega_1 t + \phi_1) + A_2 \cos(\omega_2 t + \phi_2), \quad and \tag{10}$$

$$\theta_b = \theta_{b1} + \theta_{b2} = A_1 \cos(\omega_1 t + \phi_1) - A_2 \cos(\omega_2 t + \phi_2). \tag{11}$$

The four constants A_1, A_2, ϕ_1, and ϕ_2 are chosen to satisfy the initial conditions.

Unless this system is in a single mode, some energy is transferred back and forth between the modes at a frequency called the beat frequency. The energy transfers are complete when the two modes are present in equal amounts.To achieve this take $A_1 = A_2 = A$. Also take $\phi_1 = \phi_2 = 0$. For these particular initial conditions the motions become

$$\theta_a = A \cos \omega_1 t + A \cos \omega_2 t \quad and \tag{12}$$

$$\theta_b = A \cos \omega_1 t - A \cos \omega_2 t. \tag{13}$$

Note that for these initial conditions, at t=0, $\theta_a = 2A$ and $\theta_b = 0$. Think for a moment about what this looks like.

Using standard trigonometric formulae, Eqs. 12-13 become

$$\theta_a = 2A \cos\left((\frac{\omega_2 - \omega_1}{2})t \right) \cos\left((\frac{\omega_1 + \omega_2}{2})t \right) \quad and \tag{14}$$

$$\theta_b = 2A \sin\left((\frac{\omega_2 - \omega_1}{2})t \right) \sin\left((\frac{\omega_1 + \omega_2}{2})t \right). \tag{15}$$

Defining the beat frequency ω_b and the average frequency ω_{av} as

$$\omega_b = \frac{\omega_2 - \omega_1}{2} \quad and \quad \omega_{av} = \frac{\omega_1 + \omega_2}{2}, \tag{16}$$

Eqs. 14-15 can be written as

$$\theta_a = [2A \cos \omega_b t] \cos \omega_{av} t \quad and \quad \theta_b = [2A \sin \omega_b t] \sin \omega_{av} t. \tag{17}$$

If the coupling is weak between the pendulums, $\omega_b \ll \omega_{av}$. In Eq. 17 the two quantities in square brackets act as slowly varying amplitudes for the faster varying $\cos \omega_{av} t$ and $\sin \omega_{av} t$ factors. Due to the phase difference between these two amplitudes, one pendulum has maximum amplitude when the other is at rest. The energy goes back and forth at the beat frequency.

3 Apparatus and Setup

The apparatus is shown in Figure 1. Mount two of the rods vertically to the table using the bench clamps. Slide the two rotary motion sensors onto the third rod, and attach it horizontally to the two vertical rods using the rod clamps. Attach the masses to the ends of each mini-rotational rod. Then attach each mini rod to the pulley of each sensor to make it a pendulum. Use the rubber band to connect the two rods, coupling the two pendulums.

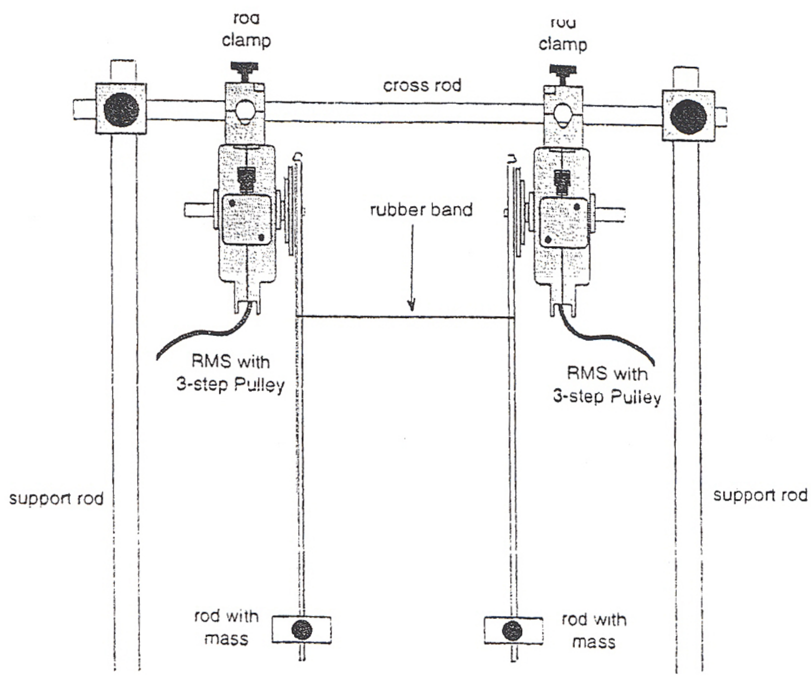
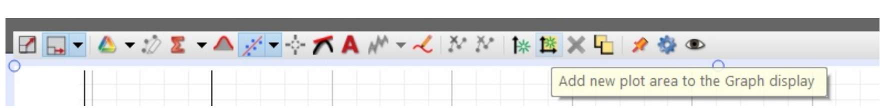


Figure 1: Coupled Pendulum Apparatus

- Plug the left rotary motion sensor in channels 1 and 2 and the right rotary motion sensor in channels 3 and 4. Program the interface for the two rotary motions sensors.
- In Hardware Setup window, click on the gear icon that lies along the first rotary motion sensor. Go to the resolution row, and choose 1440 divisions per rotation. Do this for the second rotary sensor.
- Adjust the sample rate for both sensors. This option is found on the bottom of the set up window. **What do you find works well to track the motion of the pendulums?**
- Open the graph display by dragging its icon to the center of the white screen.
- Add an additional plot to the graph by clicking on add a plot button



- For the first plot, under Select Measurements, choose Angle Ch. 1 + 2. For the second plot, choose Angle Ch. 3 + 4.
- Click on the gear icon on the top of the graph. Select Future Data Appearance, and un-check Show Connecting Lines **for both plots**. You will probably find the display easier to read without the data points.

4 Experiments

4.1 Matching the Frequencies

Remove the rubber band and measure the frequency of each pendulum. Adjust the position of the mass on the right pendulum until its frequency is as close to that of the left pendulum as you can make it. Fit a sine curve to an oscillation or two using Capstone and determine the frequencies of each pendulum more accurately. (Note that Capstone sometimes gives the *period*.) **What are the units of the frequency given by Capstone for the sine wave fit? What is the systematic uncertainty on the values of the fit? How do you determine them?** Once both pendulums are in sync, measure the distance of each mass relative to its pivot point.

4.2 Symmetric Mode

Both pendulums should now be identical. The common frequency for both pendulums is the normal mode frequency for the symmetric mode (both pendulums swinging in phase with the same amplitude). Replace the rubber band (it will remain on the apparatus for the following experiments, unless told otherwise). Measure the symmetric mode frequency. Why is the frequency of the symmetric mode identical to the common frequency of the two pendulums when the rubber band is removed?

4.3 Anti-Symmetric Mode

Now measure the frequency of the antisymmetric mode (both pendulums swinging with *opposite* phase with the same amplitude). **Is this frequency larger or smaller than the symmetric mode? Why? Can you measure K, the spring constant of the coupling? How? Does the formula for ω_2^2 agree with your measurements?**

4.4 Beats

Start only one pendulum swinging, with the other starting at $\theta = 0$, and take data. This excites both modes equally as discussed in the theory section. Observe the system go through a few "beats" and compare your data for ω_b and ω_{av} to the predictions of Eqs. 17. A printout of the pattern might be nice to include. When both modes are excited equally and each pendulum alternately oscillates and stops, there are moments when both pendulums have about the same amplitude. **How is it that one pendulum is decreasing in amplitude and the other is increasing in amplitude?**

4.5 Varying the Coupling

Moving the rubber band up or down will decrease or increase the coupling. **Why?** Adjust the strength of the coupling several times and measure the two normal modes (symmetric and anti-symmetric) and the beat pattern frequencies. The beat frequency is given by Eq. 16 in terms of the two normal mode frequencies which you have now measured. **How does each calculated beat frequency compare to your experimental results?**

4.6 Arbitrary Initial Conditions

Starting one pendulum oscillating while the other is at rest is a very special initial condition that excites both modes equally. Try a few other initial conditions, measure what happens, and compare your results to theory.

4.7 Inadvertent Coupling

Remove the rubber band. Start one pendulum swinging. **Is it still coupled to the other pendulum? What is the beat frequency? What can you do to adjust the strength of the remaining coupling?** Try finding some ways to adjust the coupling and discuss why they work the way they do.

4.8 The Two Pendulums Not Identical

Adjust one of the pendulum masses a bit so the frequencies are a bit different and measure them. Put the rubber band back on. Start one pendulum swinging and measure what happens. We have now violated one of our assumptions under which we derived our theoretical description of the motion. **Can you extend the theory to describe this more general motion?**

5 Your own experiment

Try something new with this setup! You can extend the apparatus if needed, or use it in different ways. Or experiment with some other related system with coupled normal modes! Describe your experiment in detail, show your measurements, and compare to theoretical expectations if possible.

6 Conclusions

Discuss your experiments and the results. Question: **What is another system that is described by normal modes? Can you make an analogy between it and the coupled pendulums?**