$S$-DUALITY in $N=4$ $D=4$ SYM

WORLD-VOLUME ACTION OF $N$ $D=3$ BRAVE IN IIB

\[ S_{D3} = \frac{M_5^2}{8\pi} \int d^4 \sqrt{-g} \left( -g^{\mu \nu} + \frac{1}{g_s} \int d^4 x \text{Tr} F_{\mu \nu} F_{\mu \nu} + \ldots \right) \]

\[ \mathbb{R}^4 \]

\[ U(N) \text{ YANG-MILLS in 4-D} \]

\[ = \text{SUPERSYMMETRIC COMPLETION.} \]

$D=3$ BRAKE IS SELF-DUAL UNDES $g_s \rightarrow \frac{1}{g_s}$

FROM D3 ACTION : $g_s = g_{\text{YM}}$

$g_s \rightarrow \frac{1}{g_s}$ $g_{\text{YM}} \rightarrow \frac{1}{g_{\text{YM}}}$ (STRONG-WEAK COUPLING TRANSFORMATION i.e. DUALITY).

HOW DUALITY ACTS ON D3 FIELDS $A_\mu$ ETC. ?

SIMPLE CASE 1 D3

\[ S_{D3} = \frac{1}{g_s} \int F \wedge * F + \int F \wedge dA \]

\[ \mathbb{R}^4 \]

EQUATION OF MOTION :

\[ F = \frac{1}{g_s} * F + dA = 0, \quad A : \quad dF = 0 \quad \text{ED} \quad F = dA \]
1) Substitute $F = dA$ in action

$$S_{D3} = \frac{1}{g_s} \int dA \wedge * dA$$

usual Yang Mills U(1) Lagrangian.

2) Solve $F$ equation:

$$-\frac{1}{g_s} \wedge F = d\tilde{A}$$

$$S_{D3} = g_s \int d\tilde{A} \wedge * d\tilde{A}$$

$A =$ potential, $\tilde{A} =$ dual potential

since $d\tilde{A} = * F$

Notice: the Poincaré duality

$$F \rightarrow \tilde{F} = -\frac{1}{g_s} * F$$

sends $g_s \rightarrow \frac{1}{g_s}$

for a single D3, $S$-duality of IIB induces an electric-magnetic duality.

- What happens for $N$ D3 branes?
MASSLESS STATES: \[ A^i_\mu, \lambda^i_{a j}, X^{i i} \quad i = 1, \ldots, N \]
\[ \text{spinor index in 4-d} \]
\[ \gamma = 1, \ldots, 4 \quad \text{in 4 of } SO(4) \]
\[ \alpha = 6 \quad \text{in } SO(6) \]
\[ \equiv [I,J] \quad \text{of } SO(4). \]

THEY COME FROM OPEN STRINGS

STRINGS FROM i TO j BRAKE
(DISTANCE d_{ij})

STATES \[ A^i_\mu, \lambda^i_{a j}, X^{i i} \] HAVE MASS \[ M_s d_{ij}. \]

ALL THIS IS IN PERFECT AGREEMENT WITH FIELD THEORY.

\[ N = 4 \quad 4d \text{ LAGRANGIAN:} \]
\[ \mathcal{L} = \frac{1}{g_s^2} \text{Tr} \left[ \frac{1}{4} F_{\mu \nu} F^{\mu \nu} + M_s^4 D_\mu X D^{\mu} X + M_s^2 V(X) + \right. \]
\[ \left. + \text{fermions} \right] \]

\[ V(X) = [X^A X^B] [X^B X^A]: \quad V = 0 \Rightarrow [X^A X^B] = 0 \]
Thus \( X^A \in \text{Cartan of } U(N) \equiv U(1)^N \)

\( X^A \) can be diagonalized simultaneously

\[
X^A = \begin{pmatrix}
X^{A1} \\
\vdots \\
X^{A_{NN}}
\end{pmatrix}
\]

\( X^A \) in adjoint of \( SU(N) \). Mass of \( A_{\mu}^{ij} \)

\[
M_{ij} = M_s \sqrt{\sum_{A=1}^{6} (X^{ij}_A - X^{ij}_A)} = M_s^2 d_{ij}
\]

\( M_s^2 X^A = \Phi^A = \text{canonically normalized Higgs} \)

Mass of vector \( = \sqrt{\sum_{A} \langle \Phi^A \rangle \langle \Phi^A \rangle} \)

\( \text{This state is charged under both } i\text{-th and } j\text{-th } U(1) \)

\( U(1) \quad U(1) \)

Fundamental string:

\( \text{It is a state charged under the } U(1) \text{ of D3 brane.} \)}
CAN A F-STRING END ON D3?

D3 SPANS COORDINATES 0 1 2 3
          " " " 
X¹ X² X³ 4 = X⁴

COORDINATES 5, 6, 7, 8, 9 : y¹ - y⁵.

ACTION OF II B WITH F-STRING

\[ S_{\Pi B} + S_F = \frac{1}{2} \int H \wedge \star H + \int \Theta(x^4) \delta(y) \wedge \delta(x) \wedge B \]

\[ H = dB \]

B EQ. OF MOTION \[ d \star H + \Theta(x^4) \wedge \delta(y) \wedge \delta(x) = 0 \]

INCONSISTENT! \[ d d = 0 \neq \Theta(x^4) \wedge \delta(y) \wedge \delta(x) = 0 \]

SOLUTION: \[ S_{D3} \] GRANNE ALSO CONTAINS B.

\[ S = S_{\Pi B} + S_F + S_{D3} \]

\[ S_{D3} = \frac{1}{2} \int \Theta(x^4) \wedge \delta(y) \wedge \star \wedge \star \]

\[ F = dA - B \]

B EQ. OF MOTION \[ d \star H - \Theta(x^4) \wedge \delta^3(x) \wedge \delta(y) + \\
+ \delta(x^4) \wedge \delta(y) \wedge \star F = 0 \]

d or c.o.m. : \[ d \star F = \delta^3(x) \]
ON THE WORLD-VOLUME OF THE D3 BRAVE THE F-STRING APPEARS AS AN ELECTRICALLY CHARGED STATE.

- TO HAVE ELECTRIC-MAGNETIC DUALITY WE NEED ALSO MAGNETICALLY CHARGED STATES

- D3 BRAVE COUPLES TO 2-FORM RR FIELD $C_2$:

$$\int \delta(x^4) \wedge \delta^5(y) \wedge \left[ C_4 + C_2 \wedge F + \frac{1}{2} C_0 \wedge F \wedge F \right]$$

- COLLECT ALL TERMS THAT DEPEND ON $C_2$:

- $S_{\Pi B}$:

$$\frac{1}{2} \int \tilde{H} \wedge \star \tilde{H} \quad \tilde{H} = dC_2$$

- $S_{D1}$:

$$\int \Theta(x^4) \delta^5(y) \wedge \delta^3(x) \wedge C_2$$

- $S_{D3}$:

$$\int \delta(x^4) \delta^5(y) \wedge C_2 \wedge \tilde{F}$$

$C_2$ E.O.H.:

$$d \star \tilde{H} + \Theta(x^4) \delta^5(y) \delta^3(x) + \delta(x^4) \delta^5(y) \wedge \tilde{F} = 0$$

$D1$ STRING APPEARS AS MAGNETICALLY CHARGED STATE ON WORLD-VOLUME OF D3.
\[ \text{Mass} = \frac{M_s^2}{8s} \delta_{i,j} = \frac{1}{g_s^2} \left| \Sigma_k \phi^A_k \right|^2 \]

\[ M_{\text{electrically-charged state}} = g_s^2 \]
\[ M_{\text{magnetically-charged state}} = \frac{1}{g_s} \]

**E-M duality (also called S):** \( g_s \rightarrow \frac{1}{g_s}, s \rightarrow \bar{s} \).

More generally, \( SL(2, \mathbb{Z}) \) of type IIB induces \( SL(2, \mathbb{Z}) \) of \( N=4 \) SYM that sends state with electric charge \( p \) and magnetic charge \( q \) into \( (\alpha p + c q, \beta p + d q) \)

\[
\begin{vmatrix}
\alpha & b \\
c & d
\end{vmatrix} = 1 \quad (q, b, c, d) \in \mathbb{Z}.
\]

**S-duality of \( N=4 \) SYM is a consequence of S-duality of IIB.**

**Last thing to check:** Electric multiplet is a vector multiplet of \( N=4 \):

\[
\left( \begin{array}{c}
1 (1), \\
4 (\frac{1}{2}), \\
5 (0)
\end{array} \right)
\]

**Multiplicity**

**Spin**

What is the magnetic multiplet?
D3 preserves $\frac{1}{2}$ SUSY:

$$Q_d = \Gamma^7 \Gamma^6 \Gamma^3 \Gamma^7 \Gamma^8 \Gamma^9 \tilde{Q}_d$$  \hspace{1cm} (1)

D1 preserves $\frac{1}{2}$ SUSY:

$$Q_d = \Gamma^3 \Gamma^2 \Gamma^1 \Gamma^5 \Gamma^6 \Gamma^7 \Gamma^8 \Gamma^9 \tilde{Q}_d$$  \hspace{1cm} (2)

D1 + D3 preserve $\frac{1}{4}$ SUSY

From (2)

$$\tilde{Q}_d = \Gamma^9 \Gamma^8 \Gamma^7 \Gamma^6 \Gamma^5 \Gamma^3 \Gamma^2 \Gamma^1 \Gamma^0 \Gamma^1 \Gamma^2 \Gamma^3 \Gamma^4 \Gamma^5 \Gamma^6 \Gamma^7 \Gamma^8 \Gamma^9 \tilde{Q}_d$$

Multiply by $\Gamma^2 \Gamma^5 \Gamma^6 \Gamma^7 \Gamma^8 \Gamma^9$

From (1)

$$Q_d = \Gamma^7 \Gamma^6 \Gamma^5 \Gamma^4 \Gamma^3 \Gamma^2 \Gamma^1 \Gamma^0 \Gamma^1 \Gamma^2 \Gamma^3 \Gamma^4 \Gamma^5 \Gamma^6 \Gamma^7 \Gamma^8 \Gamma^9 Q_d$$

8 real SUSY = 4 complex SUSY survive

( $\Gamma^9 \Gamma^8 \Gamma^7 \Gamma^6$ eigenvalue +1)

Write these SUSYS as $q_d^1$, $q_d^2$

\[ \begin{array}{c|c|c|c|c|c|c|c|c} \hline
   & q_d^1 & q_d^2 \\
\hline
   & & & & & & & & \\
\hline
   & & & & & & & & \\
\hline
\end{array} \]

2 = 1, 2 = Weyl spinor index

In QCD

Multiplet: 10, $\exp Q_d^\dagger Q_d^{(c)} 10$, $\exp Q_d^\dagger q_d^{(c)} 10$

5 spin 0

$\bar{q}_d q_d^{(c)} 10$ $\leftrightarrow$ 1 spin 1

$\exp q_d^{(c)} q_d^{(c)} 10$, $q_d^1 10$, $q_d^2 10$, $\exp q_d^{(c)} q_d^{(c)} 10$

4 spin 1/2
SEIBERG-WITTPEN DUALITY FROM IIA ↔ M THEORY DUALITY

A) PERTURBATIVE CONSTRUCTION \((gs \ll 1)\)

\[
\begin{array}{ccc}
\text{D4} & \text{X} & \text{NS5} \\
\text{NS5} & \text{X}^6 = 0 & \text{NS5} \\
\text{D4} & \text{X}^6 = L &
\end{array}
\]

NS5 spans 0 1 2 3 4 5
D4 spans 0 1 2 3 6

WORLD-VOLUME FIELDS OF NS5 : \( b_{\mu}^+, \phi, X^6, X^9 \)

\[ \text{I} \text{SELF-DO} \mu \text{AL} \ h = \text{d}b \ \text{IN} \ \text{6-D} \]

IMPOSSIBLE : F STRING ENDING HAS A CHARGE, BUT ON NS5 WORLD-VOLUME THERE IS NO VECTOR TO CARRY IT.

D2 ENDING ON NS5 IS A STRING ON THE WORLD-VOLUME OF NS5. IT COUPLES TO \( b_{\mu}^+ \).

D4 CAN END ON NS5. ON WORLD-VOLUME OF NS5 IT IS A 3-BRANE. IT MUST COUPLE TO A
4-form $A_4$. $A_4$ exists as it is the 6-D dual of $\phi$:
\[ \ast d\phi = d A_4. \]

Fields on world-volume of D4 suspended in between 2 NS5s.

\[ A_m = X^4, X^5, X^7, \ldots X^9 \]
\[ \uparrow \quad \downarrow_{0,1,2,3,6} \]

Boundary conditions on these fields are: $X^7(0) = X^7(L)$ etc.

D4 has finite length in $X^6: 0 \leq X^6 \leq L$

Expand in Fourier series in $X^6$:
\[ X^7, \ldots X^9 (X, X^6) = \sum_{\overset{n \neq -1}{0,1,3}} \infty \sim (X^6/L) \pm \pm \sum_{n=1}^{\infty} X^7 (x) \sin \left( \frac{X^6}{L} \pi \right) \]
\[ \uparrow \quad \downarrow \quad \text{no zero modes.} \]

At energy scales $E \ll \frac{1}{L}$ only zero modes survive: no $X^7, \ldots X^9$.

Notice: Coordinate $X^6$ is longitudinal for D4, transverse for NS5. Both $\partial_x X^6 = 0$, $\partial_6 X^6 = 0$.

Vertex for $A^6$ is $\partial_x X^6 \exp(i k x) = 0$.

$A^6(x, 0) = A^6(x, L) = 0$ no zero modes for $A^6$.

Degrees of freedom at $E \ll \frac{1}{L}$ $A_{\alpha}^{(0)}(x)$, $X^4(x)$, $X^5(x)$.
For $E \ll \frac{1}{L}$, \[ S_{D4} = \frac{M_5 L}{g_5^2} \int d^4 x \text{Tr} \ F_{\mu \nu} F^{\mu \nu} + \ldots \]

\[ \frac{M_5 L}{g_5^2} = \frac{1}{g_{YH}^2} \quad \text{(at scale } \mu = \frac{1}{L}) \]

\[ Q_i \bar{Q}_j = 3 \ 2 \text{ SUSY} \quad \text{NSS break them to 16 SUSY} \]

\[ D4 \text{ break 16 SUSY to } 8 \quad (\equiv N=2 \ D=9 \text{ SUSY}) \]

**MULTIPLETT:**

\[ A_\mu, \ V = X^4 + iX^5, \ \Omega \]

**VECTOR MULTIPLETT OF** $\mathcal{N}=2$.

**4-D GAUGE GROUP IS** $SU(2)$, **not** $U(2)$

**U(2) has 2 complex moduli - they are here.**

**The position of the D4 inside NSS**

\[ X_5 \]

\[ \begin{array}{c}
  \bullet Q_1 \\
  \bullet Q_2 \\
  \rightarrow X_4
\end{array} \]

\[ \text{D4 is massive - it bends} \]

\[ \text{NSS} \]

\[ \text{D4} \rightarrow \]

\[ \sum_i \delta \left( V - Q_i \right) \]

\[ \Delta X^6 = \sum_{i=1}^2 \sum_{i=1}^2 \delta \left( V - Q_i \right) \]

**Solution:** \[ X^6 = \sum_i \left[ g_{i} - |Q_i| \right] \]
KINETIC TERM FOR NS5 CONTAINS:
\[
\int d^4x \int d^4\sigma \partial_\mu \alpha^6 \partial^\mu \alpha^6 = \int d^4x d^4\sigma \left| \sum_i \frac{1}{\sqrt{-g}} \partial_\mu \alpha_i \right|^2
\]
\[
\sqrt{-g} \rightarrow \alpha \quad \left| \sum_i \frac{1}{\sqrt{-g}} \partial_\mu \alpha_i \right|^2 \rightarrow \frac{1}{\alpha} \left| \partial_\mu \sum \alpha_i \right|^2
\]

FINITE KINETIC TERM ONLY IF \( \partial_\mu \sum \alpha_i = 0 \)
\( \Rightarrow \sum \alpha_i = \) constant.

CENTER OF MASS OF SYSTEM OF 2-D4 BRANES IS NON-DYNAMICAL. \( \alpha_1 - \alpha_2 \) IS THE ONLY DYNAMICAL MODULUS.

1 MODULUS IS GAUGE GROUP IS SU(2).

QUESTION: WHERE ARE THE MONOPOLES?

\[
\begin{array}{c|c}
\text{D2} & \text{D4} \\
\hline
\text{D4} & \text{NS5 SPANS COORDINATES 0 1 2 3 4 5} \\
\text{D2} & \text{D4} \\
\text{NS5} & \text{NS5}
\end{array}
\]

\[
\text{Mass of D2} = \frac{1}{g_5} M_5^3 L D = \frac{M_5^3 \langle \phi \rangle}{g_5}
\]

ON. WITH MASS OF MONOPOLE \( \frac{1}{g_4} \frac{\langle \phi \rangle^2}{L} \)

IS THIS D2 MAGNETICALLY CHARGED UNDER THE \( U(1) \) OF THE D4?
ACTION: \( S_{IIA} + S_D^2 + S_4^D = \int G_4 \wedge \ast G_4 \)
\( + \int \theta(x^9) \delta^3(x) \wedge \delta^4(y) \wedge C_3 + \int \delta(x^9) \delta^4(y) \wedge F \wedge C_3 \)

\( C_3 \) E.O.M.: \( d \ast G_4 - \theta(x^9) \delta^3(x) \wedge \delta^4(y) - \delta(x^9) \wedge \delta^4(y) \wedge F = 0 \)

OF E.O.M.: \( dF = + \delta^3(x) \leq \text{magnetic charge} \)

SUSY: \( 32 \xrightarrow{\text{NS5}} 16 \xrightarrow{\text{D4}} 8 \xrightarrow{\text{D2}} 4 = \mathcal{N}=1 \ 4\text{-d susy} \)

SURVIVING SUSY: \( Q_\alpha \) (WEYL SPINOR)

\( \leq 1, 2 \)

MULTIPLETS:
\( |q\rangle, |Q_\alpha q\rangle, |Q_1 Q_2 l q\rangle \)

\( q = \text{charge of monopole} \)

+ C.P.T. \( |l-q\rangle, |Q_\alpha l-q\rangle, |Q_1 Q_2 l-q\rangle \)

4 SPIN ZERO + 2 SPIN \( \frac{1}{2} \) (HYPERMULTIPLE OF \( \mathcal{N}=2, D=4 \))

\[ \begin{array}{c}
\text{D4} \\
\hline
\text{D4}
\end{array} \]

DESCRIBES A \( SU(2) \) \( \mathcal{N}=2 \) SUPER YANG MILLS.

WE GAVE THE PERTURBATIVE DESCRIPTION HERE. WHAT IS THE NON-PERTURBATIVE ONE?
B) NON-PERTURBATIVE DESCRIPTION

\[ g_5 \to \infty \quad M_{5L} \to \infty \quad \text{with} \quad \frac{M_{5L}}{g_5} = \frac{1}{g_{YM}} = \text{constant} \]

\[ g_5 \to \infty \quad \Rightarrow \quad \text{A} \to \text{M THEORY} \]

RELATION \( \frac{M_{5L}}{g_5} = \frac{1}{g_{YM}} \) \text{ IS PERTURBATIVE}

NON-PERTURBATIVELY: \( (g_5, M_{5L}) \sim 2 \text{ Parameters} \)

But SYM depends on \( g_{YM} \) only.

One can send \( g_5 \to \infty \quad M_{5L} \to \infty \) keeping \( g_{YM} = \text{const.} \)

- How does one lift \( \begin{array}{c} D4 \cr D4 \end{array} \) \( \text{NSS} \quad \text{NSS} \) to D=11 SUPERGRAVITY

RECALL: IN M-THEORY DESCRIPTION OF D=11-TH
COORDINATE \( X^{10} \) COMPACTIFIED \( X^{10} \propto X^{10} + 2\pi R \)

Both D4 and NSS come from the same M-theory object: A 5-BRAVE

DEFINE \( s = (X^6 + iX^{10})/R \quad s \simeq s + 2\pi i \)

Better to use \( t = e^{-s} \)

M 5-BRAVE CONFIGURATION IS \( \mathbb{R}^{(3,1)} \times \Sigma \)

\( \Sigma \) is a RIEMANN SURFACE in \( \mathbb{C}^2 (t, u) \)
M5 has positive tension; $\Sigma$ is a minimal surface in $\mathbb{C}^2$.

Described by holomorphic curve $F(t, v) = 0$

At $v = \text{const}$, it must have 2 roots (the position of the NSS branes)

$$F = A(v) t^2 + B(v) t + C(v)$$

At $t = \text{const}$, it must also have 2 roots (the position of the D4 branes)

$A$, $B$, $C$ quadratic in $v$.

If $C(v)$ has root $C(v) = 0$, then $t = 0$

This means that there is a D4 at $t = 0$

IE. at $e^{-(x^6 + ix^{10})/\lambda} = 0$, $x^6 = +\infty$

Not our configuration.

Thus $C = \text{const}$ (=1 by rescaling).
IF $A(v) = 0$ THEN $t = \infty$ ($x^6 = -\infty$) IS A ROOT.

CONFIGURATION:

$$\partial_y |_{NS5}$$

So: $A(v) = 1$ WITH A TRANSLATION IN $v$

$$F(t, v) = t^2 + (v^2 + u) t + 1$$

$$\downarrow \text{ const.}$$

$$t^2 + (v^2 + u) t + 1 = 0$$ IS THE "AUXILIARY" SURFACE IN THE SEIBERG-WITTEN EQUATION OF $N = 2$ SYM. HERE IT IS A REAL SURFACE: PART OF THE WORLD VOLUME OF THE $M5$ BRAVE.

BRANCH POINTS OF $F$: $(v^2 + u)^2 - 4 = 0$

$$v^2 = \pm 2 - u$$

THE SURFACE BECOMES SINGULAR WHEN THE BRANCH POINTS COINCIDE:

1) $v^2 = 2 - u = 0 \Rightarrow u = 2$
   $v^2 = -2 - u = 0 \Rightarrow u = -2$
   AND $v^2 = \infty$, $u = \infty$.

$u$ PARAMETRIZES A FAMILY OF SURFACES. $u$ IS $S^2 - 3$ POINTS (AS IN SEIBERG-WITTEN)
IN \( H \) THEORY ELECTRICALLY & MAGNETICALLY CHARGED STATES COME BOTH FROM THE SAME OBJECT: THE M\(_2\) BRANE ENDING ON \( \Sigma \).

\[
\text{AREA OF M}_2 \propto \int_{\text{world-volume of M}_2} \omega
\]

\( \omega = \text{holomorphic 2-form} \quad \frac{dt \wedge dv}{t} = dx \) (locally)

\[
\int_{\partial D} \omega = \int_{\partial D} \frac{dt \wedge dv}{t} = \int_{\partial D} \log t(v) dv = \int_{\partial D} v \frac{dt}{t}
\]

\( \text{AREA DEPENDS ONLY ON } \partial D. \)

- CHOOSE A BASIS OF HOMOLOGY CYCLES IN \( \Sigma \):

\( \gamma_a, \gamma_b \quad \gamma_a \cap \gamma_b = 1 \) \( \text{(intersection number 1)} \)

\( \text{Call} \quad \int_{\gamma_a} \gamma + dv = a(u) \quad \int_{\gamma_b} \gamma + dv = a_0(u) \)

STATE WITH \( \partial D = p\gamma_a + q\gamma_b \quad p, q \) integers

HAS MASS

\[
| p \int_{\gamma_a} \gamma + q \int_{\gamma_b} \gamma | = | p a(u) + q a_0(u) | = \text{mass}
\]

OF BPS STATES OF \( N = 2 \)!
We can also get the low-energy action of the modulus of $N=2$.

Recall that MS has world-volume 2-form $b^+$:

$$db = H = *H \quad (\text{self-dual})$$

$b^+$ gives a 4-d vector:

$$H = F \Lambda (\Lambda + \Lambda\bar{\Lambda}) + *F \Lambda \Lambda (\Lambda + \Lambda\bar{\Lambda})$$

$$dF = d*F = 0 \quad (\text{eq. of motion})$$

$$d\Lambda = d\Lambda\bar{\Lambda} = 0.$$  

$\Lambda$ is a normalizable 1-form on $\Sigma$.

$$\Lambda = \frac{2\alpha}{\partial u} \quad \alpha = \frac{v}{t} \int_{\Sigma} \quad \text{or} \quad t \neq 0 \text{ in } \Sigma$$

$$\alpha = l \int_{\Sigma} \text{function of } v \text{ only on } \Sigma$$

Compute $\frac{\partial \alpha}{\partial u}$.

From

$$t^2 + (v^2 + u) t + 1 = 0 \quad 2 \frac{\partial t}{\partial u} t + (v^2 + u) \frac{\partial t}{\partial u} = 1$$

when $t = t(v,u)$

or

$$2v \frac{\partial v}{\partial u} t + t = 0 \quad \text{when } v = v(t,u)$$
\[
\frac{\partial t}{\partial u} = \frac{-t}{zt + v^2 + u} \quad \text{or} \quad \frac{\partial v}{\partial u} = -\frac{1}{2v}
\]

Two forms for \(\frac{\partial x}{\partial u}\):

1) \(\frac{\partial x}{\partial u} = -\frac{1}{2vt}\)
2) \(\frac{\partial x}{\partial u} = -\frac{1}{zt + v^2 + u}\)

\(F(\tau, v) = 0\) \(\text{regular} \) \((u \neq \pm 2, \infty)\)

\[
\left(\frac{\partial F}{\partial t}, \frac{\partial F}{\partial v}\right) = (zt + v^2 + u, 2vt) \neq (0, 0)
\]

So wherever \(F\) is regular either \(2vt\) or \(zt + v^2 + u\) are nonzero, \(\frac{\partial x}{\partial u}\) is well defined everywhere on \(\Sigma\).

Moreover \(\lambda^2 \sim \frac{dV \Lambda d\bar{\Lambda}}{1/4}\) for \(|u| \to \infty\) \(\Lambda\) normalizable

\(\Lambda\) holomorphic: \(\ast \Lambda = \bar{\lambda}\)

Kinetic term of \(F\) :

\[
\int_{\Sigma} F \Lambda \ast F + \int \Lambda \wedge \bar{\Lambda}
\]

\(d\Lambda = 0\) so \(\Lambda = \frac{\partial u}{\partial m} h_1 + \frac{\partial \phi}{\partial u} h_2 + du\) Harmonic 1-forms on \(\Sigma\)
\[ \sum \bigwedge \Lambda = i \sum \left( \frac{\partial q}{\partial u} h_1 + \frac{\partial p}{\partial u} h_2 + d\omega \right) \Lambda \left( \frac{\partial q}{\partial u} h_1 + \frac{\partial p}{\partial u} h_2 + d\omega \right) \]

\[ = i \frac{\partial q}{\partial u} \left( \frac{\partial p}{\partial u} \right) - i \left( \frac{\partial q}{\partial u} \right) \frac{\partial p}{\partial u} \]

Since:
\[ \sum h_1 \wedge d\omega = \sum h_2 \wedge d\omega = 0 \]
\[ \sum h_1 \wedge h_2 = 1 \]

\[ i \frac{\partial q}{\partial u} \left( \frac{\partial p}{\partial u} \right) - i \left( \frac{\partial q}{\partial u} \right) \frac{\partial p}{\partial u} \] is the kinetic term of the low-energy effective theory as in Seiberg-Witten.

REFS:

E. Witten  \textit{HEP-TH/9503124} , \textit{HEP-TH/9703166}

J. de Boer, K. Hori, H. Ooguri and Y. Oz  \textit{HEP-TH/9711143}
INTERLUDE: BOUNDARY STATES.

\[ \text{Closed String} \]

\[ \text{Dirichlet} \]

\[ \partial_\tau X^\mu \bigg|_{\tau=0} = 0 \quad \mu = 0, \ldots, \rho \]

**Homework:** Find these boundary conditions by interpreting the closed-string diagram as an open-string 1-loop diagram.

ON DIRICHLET COORDINATES:

\[ X^A 1B \rangle = y^A 1B \rangle \quad A = 0, \ldots, D \]

Mode Expansion:

\[ \partial_\tau X^\mu \bigg|_{\tau=0} = 2 \alpha' p^\mu + \sqrt{2} \alpha' \sum_{n \neq 0} (\alpha_n + \tilde{\alpha}_n) e^{i n \tau} \]
\[ X^A |z=0 = q^A + i \sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} (x^A_n - \tilde{x}^A_n) e^{2 i \pi n z} \]  \tag{117}

Equations for boundary state become:

**N-coordinates:**

\[ (x^A_n + \tilde{x}^A_{-n}) |B> = 0 \quad n \neq 0 \]

\[ p^\mu |B> = 0 \]

**D-coordinates:**

\[ (x^A_n - \tilde{x}^A_{-n}) |B> = 0 \quad n \neq 0 \]

\[ q^A |B> = y^A |B> \]

Solution:

\[ |B> = N \int D-p (q^A - y^A) \prod_{n=1}^{\infty} e^{\frac{i}{\alpha'} x_{-n} \tilde{x}^A_{-n}} |10 \otimes |00 \otimes |p=0> \]

\[ S = \text{diag}(\eta^{\mu\nu}, -\delta^{AB}) \]

This is the solution to the (linearized) closed-string equations of motion in the presence of a Dp-brane source. At large distance from the Dp, |B> must approximate the supergravity solutions we found earlier.
More precisely:  \( |\Phi(x)\rangle = \text{massless string state} \)

\( D = \text{closed-string propagator} = \int_0^1 \frac{dz}{z^2} \frac{L_0 - \frac{1}{2} L_0^2}{z^{L_0 - 1}} \)

Then:

\( \langle \Phi(x) | D \mid B \rangle \approx \Phi_s(x) \)

\( \Phi_s(x) = \text{supergravity background} \approx \frac{\text{const}}{|x^A - y^A|^{D-p-2}} \text{ far from the brane} \)

\( |\tilde{\Phi}(p^A)\rangle \approx \delta_{i,j} \frac{1}{2} \alpha_1^i \gamma_1^j \mid 0\rangle \otimes \mid 0\rangle \otimes |p^A\rangle \otimes |p^A = 0\rangle \)

\( i, j = (\mu, A) \)

\( \langle \tilde{\Phi}(p^A) | D \mid B \rangle \approx \langle p^A | \int_0^1 \frac{dz}{z^2} \frac{1}{z} \int \frac{d^{D-p}q}{(2\pi)^{D-p}} e^{i \hat{p} \cdot q^A} \int (q^A - y^A) \mid q^A \rangle \times \)

\( \text{Apply } D \text{ to } \langle \Phi(p^A) \rangle \)

\( = \text{constant} \int \frac{dp}{p} \int \frac{d^{D-p}q}{(2\pi)^{D-p}} e^{i \hat{p} \cdot q^A} \int (q^A - y^A) \mid q^A \rangle \)

\( = \text{constant} \times \frac{1}{(p^A)^2} \text{ O.K. Fourier transform in } D-p \)

\( 1 = \frac{1}{(x-y)^{D-p-2}} \)
Correspondence between stringy boundary state and supergravity solutions becomes more precise in D=10 superstrings (Ref: P. Di Vecchia, A. Liccardo, HEP-TH/9912161).

Here we performed a semi-quantitative check of the double role of Dp branes:

A) They describe open-string sectors of (super)-string theory, corresponding to their excitations.

**Important**: At low energy the open-string sector is a gauge theory.

B) They are sources for closed-string states. They are solutions of the closed-string equations of motion. In string theory, these solutions are known at linear order (boundary states) and

**Important**: In some (a) supergravity E.O.M., are reliable (when curvature of solutions is smaller than $M_5$).

**Question**: Can we use this dual role of Dp branes to better understand string theory and/or field theory?