Matrix models for M-theory

Or, fun with big matrices. The subject first came up in


was completely reinterpreted by

Banks, Fischler, Shenker, Susskind. hep-th/9610043

got subsumed (to my mind) in

Itzhaki, Maldacena, Sonnenschein, Yankielowicz. hep-th/9802042

was reviewed by

Taylor. hep-th/0101126

and may be making a comeback in

Berenstein, Maldacena, Nastase. hep-th/0202031

Outline:

1. Light-cone KINEMATICS
2. Light-cone M-theory and the BANSS conjecture
3. recovering the spectrum - gravitons, membranes, S-branes
4. recovering interactions
5. compactification (on tori, mostly)
6. some difficulties with the BANSS limit
7. the IMSY perspective
8. pp waves
Light-front kinematics

$x^0, x^1, \ldots, x^{D-1}$ Minkowski coordinates

Periodically identify $x^{D-1} \approx x^{D-1} + 2\pi k$.

Consider the sector of the theory with $N$ units of momentum around the $x^D$.

$p^{D-1} = N/k$.

For a system of free particles

$$H_{\text{particle}} = \sqrt{p^2 + m^2} = \sqrt{\left(\frac{p}{k}\right)^2 + \frac{m^2}{k^2}} + \frac{m^2}{2p^{D-1}} + \cdots$$

In the "infinite momentum frame" $N \to \infty$ with $k$ fixed,

so $p^{D-1} \to \infty$ and

$$H = \frac{p^2}{k^2} + \frac{m^2}{2p^{D-1}} + \cdots$$

It's natural to introduce light-front coordinates

$$x^+ = \frac{1}{k}(x^0 + x^{D-1})$$

$$x^+ = t = \text{light-front time coordinate}$$

Light-front Hamiltonian

$$H = \frac{\partial}{\partial x^+} = i\frac{\partial}{\partial x^0} + i\frac{\partial}{\partial x^{D-1}} = E - p^{D-1} =$$

$$= \frac{p^2}{k^2} + \frac{m^2}{2p^{D-1}},$$

looks like non-relativistic kinematics with $p^{D-1} \to \text{mass} (\to \infty \text{ as } N \to \infty)$.

The underlying $SO(0,1,D-1)$ Lorentz invariance is broken by the choice of infinite momentum frame, so the light-front Hamiltonian only has a manifest Galilean symmetry acting on the transverse momenta.
BFSS conjecture

Study Dp theory in the infinite momentum frame.

\[ \text{II-Pr Planck length } L_{pl} \]

Periodically identify \( X^\parallel = X^\parallel + 2\pi R_{\parallel} \).

Study sector with \( N \) units of momenta on \( S^1 \).

This is equivalent to IIa string theory with

\[ g_s = \left( \frac{R}{L_{pl}} \right)^{3/2}; \quad L_s = \sqrt{\frac{L_{pl}^3}{R}} \]

sector with \( N \) units of D-brane charge.

So we have a system of \( N \) DD-branes. In the IMF the D-branes are slowly moving in the transverse directions (time dilation!). This means we only need to keep the leading terms in the velocity expansion of the D-brane effective action.

This is uniquely fixed by the symmetries in the IMF frame:

- Galilean invariance in the nine transverse directions.
- 16 supercharges (the ground state in the sector \( p^0 = N/R \) is \( 1/8 \)-BFSS preserve 16 manifest supercharges in IMF).

\[ \mathcal{S} = \frac{1}{2g_s L_s} \text{Str} \left[ \partial^\alpha \bar{\psi}^i \gamma^\alpha \psi^i + \frac{i}{2} \bar{\psi}^i (\gamma^\alpha \gamma^\beta) (\gamma^\gamma \psi^j) \right] \]

\[ + \theta^T (i\theta - \frac{1}{2} [\bar{\psi}^i, \bar{\psi}^j] \theta) \]

\( \psi^i, i = 1, \ldots, 4 \) \( \times \) \( N \times N \) Hermitian matrices

\( \theta^a, a = 1, \ldots, 16 \) Grassmann matrices

This action is the dimensional reduction of the \( U(N) \) symmetric from 3+1 to 0+1 dimensions (at the gauge \( A_0 = 0 \)).

Alternatively, it's the two-derivative truncation of the super-DBI action for D-branes.
Other stringy degrees of freedom seem to decouple. For example, the 0-branes have small velocities and small accelerations, and won't emit gravitational radiation.

(That doesn't mean gravitational back-reaction is negligible - more later when we talk about IMSY.)

This leads to the BFSS conjecture:

The Yang–Mills quantum mechanics of 0-branes provides a complete description of M theory in the infinite momentum frame.
Recovering the spectrum

As a first test, let's see if we can recover the spectrum of excitations from the D-brane quantum mechanics. We ought to be able to find

11-D SU(3) x SU(2) x U(1) multiplet

membranes (coupled to G(2))

5-branes (coupled to the dual G(6))

SU(3) multiplets?

First consider a single D8-brane, with

\[ S = \frac{1}{2g_s l_s} \int d\tau d^8 x \left( \partial \Phi \cdot \partial \Phi + i \partial \Phi \cdot \bar{\partial} \phi \right) \]

(abelian so no interactions). This is the action for a single free non-relativistic particle, moving in the nine or so dimensions, with mass \( m = \frac{g_s l_s}{2} \).

Expected mass for a D8. The Grassman coordinates \( \theta_a \) give 16 real fermion zero modes,

\[ \hat{A} = \frac{1}{2m} \left( \partial \theta + \frac{1}{3} \theta \cdot \theta \right) \]

The fermion zero modes give a

\[ \mathbf{256} = 16 \otimes 16 \]

dimensional space of states, exactly the states in the M-theory SU(3) x SU(2) x U(1) multiplet.
what about gravitons with more than one unit of p\|? To make a graviton with N units of p\|, put together N DB branes in a "threshold bound state", exactly like a relative wavefunction, energy, ... is normalizable, ... 16 supercharges.

If such a bound state exists (uniquely!), it will have the right quantum n\| to correspond to a graviton with p\| = N/k\|, arise from e.m. wavefunction.

The existence of these threshold bound states has been established, but few properties are known.

Sethi + Stern, Horowitz+ Rotter
9705046 ... 9708119

Membranes?
This was the first thing people understood about M(atrix) theory (de Wit+ Heppe+ Nicolai, 1988: ... 546; see also Kaban and Taylor 9711078).
Consider setting
\[ x^1 = \frac{2\pi}{N} J^1 \quad x^2 = \frac{2\pi}{N} J^2 \quad x^3 = \frac{2\pi}{N} J^3 \]
\[ J^i = \text{generators of } N\text{-dimensional reps. of } SU(2) \]
This describes a spherical M=2-brane of radius \( r \), embedded in \( \mathbb{R}^3 \approx \{ (x^1, x^2, x^3) \}^3 \).
To see this note that
\[ (\vec{x}_1)^2 + (\vec{x}_2)^2 + (\vec{x}_3)^2 = \frac{\nu^2}{M^2} - \frac{1}{\nu^2} = \frac{\nu^2}{M^2} (1 - \frac{1}{\nu^2}) \hat{A} \]
so that the 0-branes are localized on the surface of a sphere. The coupling to the 3-form \( C_{\mu \nu \rho} \) arises from the interaction term
\[ \text{Seff} = G_{0i \mu} \text{Tr}(\vec{x}_i \vec{X}^\mu) \]
in the action for \( N \) 09-branes: \([\vec{x}_i \vec{x}_j] \to 0 \Rightarrow \text{non-zero membrane charge (at least locally).} \]

5-branes?

M-5-branes which are wrapped around the \( S^4 \) are seen as D-4-branes in IIA. These objects can be described in a manner parallel to the M-2 story.

- Collection of four
  - E. g. \( \text{Tr}(\vec{x}_i \vec{x}_j \vec{x}_k \vec{x}_l) \to 0 \)

For explicit examples describing a round spherical wrapped M-5-brane see Costello, Lee, Taylor 9712105.

It's not clear how to describe an M-5 that isn't wrapped around the \( S^4 \), in fact it may not be possible to have an unwrapped M-5 in the IIA.
To describe multiple objects in Calabi-Yau theory, one introduces block-diagonal matrices.

\[
\mathbf{X} = \begin{pmatrix}
\text{graviton} & 0 \\
0 & \text{membrane} \\
\end{pmatrix}
\]

The interactions only involve commutators:

\[
\mathcal{W}_{\text{int}} = \text{Tr} \left( \frac{i}{2} [X, \xi]^2 [X, \xi] - [X, \theta]^2 \right)
\]

So classically, the different blocks don't interact with each other. Quantum mechanically, if the objects are mutually BPS, they still don't interact.
Recovering interactions

Suppose we had two objects which weren't mutually BPS. How can we set their interaction potential? For example

\[ X^i = \begin{pmatrix} \text{graviton at } x_1 \\ \text{graviton at } x_2 \end{pmatrix} \]

potential \( V(x_1, x_2) \)?

**Matrix theory:**

Integrate out the off-diagonal blocks to compute an effective potential \( V_{\text{matrix}}(x_1, x_2) \).

You can do this explicitly at one-loop:

1st graviton described by \( F_1 = \bar{X}_1 \)

\( F_1^{ij} = i \left[ \bar{X}_1^i, X_1^j \right] \)

2nd graviton described by \( F_2 = \bar{X}_2 \)

\( F_2^{ij} = i \left[ \bar{X}_2^i, X_2^j \right] \)

\[ V_{\text{matrix}} = -\frac{5}{128} \frac{1}{1-x_1^2} \text{STr} \left\{ 24 \left( F_1 \otimes F_2 - F_1 F_2 \right)^2 \right\} \]

1st graviton exchange

\[ V_{\text{SUGRA}} = -\frac{15}{4} \frac{R^2}{x_1 - x_2} \]

\[ \frac{1}{2} T \frac{T}{2} - \frac{15}{4} \frac{R^2}{x_1 - x_2} \]

\[ \frac{1}{12} \left( \frac{1}{x_1 - x_2} \right)^4 \]

\[ \frac{M_1 M_2}{x_1 - x_2} \]

Stress tensors -- 3-form currents -- 6-form currents

Graviton exchange -- electric 3-form exchange -- magnetic 3-form exchange
Rather remarkably, with a suitable dictionary relating matrix operators to super currents, the two potentials are in precise agreement!

Kabat + Taylor - 9712135
Taylor + van Raamsdonk - 9812239

\[ T \mu = \frac{1}{R} Tr \left( \frac{i}{2} \tilde{E} \right) \]

\[ \left[ \tilde{E} \right] = \left[ \tilde{E} \right] \left( \tilde{E} \right) \]

Compactification

*Matrix* theory describes M-theory in a Minkowski background in the infinite momentum frame, in terms of the dynamics of \( N = 8 \) O-branes in flat space.

Natural guess for compactified M-theory: let the O-branes move on a compact space.

Unfortunately, the O-brane action in a curved background isn't very well understood. Some progress has been made in compactifying on tori.

\( N \) D-O-branes T-dual \( N \) D-p-branes

on \( T^p \) on dual \( T^p \)

giving \((p+1)\)-dimensional SYM on \( T^p \) (Taylor, 9811042)

This makes some properties clear, in particular some of the T-duality groups of toroidally compactified M-theory become manifest. However even compactification on \( T^6 \) breaks down for large \( p \); more dramatically, on \( T^p = 7 \), the dual D-p-branes are codimension 2 and have fixed a deficit angles that limit \( N \leq 24 \).

No \( N \to \infty \) limit is possible.
Difficulties with BFSS limit

The reason Matrix theory is hard is that the 0-brane quantum mechanics is strongly coupled. To describe M-theory in the IMF with \( R \gg \ell_{\text{Pl}} \), we need

\[ N \to \infty \quad \frac{\alpha'}{N} = 3 \ell_{\text{Pl}} = \left( \frac{R}{\ell_{\text{Pl}}} \right)^{3/2} = R \] \( \text{fixed} \)

(or even \( \frac{\alpha'}{N} \to \infty \) to decompactify \( S^3 \))

So there's very little control over the 0-brane quantum mechanics, aside from the BPS spectrum. Even the spectacular matching of potentials we saw earlier can be traced to SUSY non-renormalization theorems which only apply at one loop (see [Aharony, Sen])

There may be multiloop non-renormalization theorems for some terms in the action

more concretely, as \( N \to \infty \) all states become small perturbations on the (very poorly understood) threshold bound state of \( N \) 0-0-branes
The BFSS conjecture can be "derived" by studying the near horizon geometry of $N=6$ D=0-brane back-reaction (Itzhaki et al. 9802048).

In the BFSS limit one finds that the dual near-horizon geometry is a region of 11-dimensional flat space. Thus BFSS can be regarded as a non-conformal variant of AdS/CFT.

PP waves

Who knows; this may all come back. Berkooz et al. 0202021 studied M-theory on $AdS_4 \times S^7$ in a sector with $T \to \infty$ units of K.K. momentum on the $S^7$. They write down the corresponding $M(\infty)$ model, a $U(5)$ 5dM theory. Because of the extra terms in their matrix action, arising from the curved "PP wave" backgrounds, the BMN matrix model seems to be much easier to study than BFSS.