1.1 Poisson Distribution

a) Divide the time $t$ in small increments $dt$. For each of these increments the probability of a collision is $dt/\tau$ and hence the probability of no collision is $1 - dt/\tau$. The number of intervals of length $dt$ are $t/dt$. The probability of not having a collision is then the product of the probabilities for each of the intervals

$$P_{\text{no coll.}}(t) = \left(1 - \frac{dt}{\tau}\right)^{t/dt} = \frac{1}{\tau} \ln(1 - dt/\tau) = e^{-(dt/\tau + O(dt^2))} = e^{-t/\tau}$$  \hspace{1cm} (1)

Alternatively, probability of no collision $P_{\text{no coll.}}(t)$ can be calculated as follows. In order for the particle to not have a collision during the last $t + dt$ seconds it must not have collided in the last $t$ seconds also it must not collide in the last part $dt$. (Note, time $t$ is previous time).

$$P_{\text{no coll.}}(t + dt) = \left(1 - \frac{dt}{\tau}\right) P_{\text{no coll.}}(t)$$  \hspace{1cm} (2)

which gives

$$\frac{dP_{\text{no coll.}}}{dt}(t) = \frac{P_{\text{no coll.}}(t + dt) - P_{\text{no coll.}}(t)}{dt} = -\frac{1}{\tau} P_{\text{no coll.}}(t).$$ \hspace{1cm} (3)

Solution gives $P_{\text{no coll.}}(t) = e^{-t/\tau}$.

b) The probability that the time between two successive collisions of an electron is in the interval $[t, t + dt]$ is obtained as follows. Let $t = 0$ be the time directly after the previous collision. In order to have a collision in the interval $[t, t + dt]$ it must first not collide in the interval $[0, t]$ and then collide in the interval $[t, t + dt]$. The probability for this to happen is now obtained as the product of the probability of these events

$$\frac{dt}{\tau} e^{-t/\tau}. \hspace{1cm} (4)$$

The probability of having a collision in the time interval $t$ to $t + dt$ is usually written $P(t)dt$ where then

$$P(t) = \frac{1}{\tau} e^{-t/\tau} \hspace{1cm} (5)$$

is the probability density function.

c) We now want to look at the time since the last collision averaged over all electrons. If we assume that we have $N$ electrons we will on average have $Ne^{-t/\tau}$ electrons which did not collide during the preceding time $t$. Similarly we have $Ne^{-(t+t dt)/\tau}$ that did not collide during the preceding time $t + dt$. The difference in number comes from the fact that some electrons actually did collided during the time $[-(t + dt), -t]$ ($t = 0$ is now), the number of such electrons is

$$Ne^{-t/\tau} - Ne^{-(t+t dt)/\tau} = -N \frac{e^{-(t+t dt)/\tau} - e^{-t/\tau}}{dt} dt = -N \left(\frac{e^{-t/\tau}}{\tau}\right) dt$$

$$= N \frac{e^{-t/\tau}}{\tau} dt \hspace{1cm} (6)$$
This is hence the number of electrons that collided in a small interval of time $[-(t+dt),-t]$ in the past. Now we want to find time since last collisions average over all electrons:

$$\langle t \rangle = \frac{1}{N} \int_0^\infty t N e^{-t/\tau} dt = \tau. \quad (7)$$

d) Subtle difference between this and the last problem! The previous problem was an average over all electrons. Here we just consider a single electron. For a randomly picked electron it has a collision in the interval $[t, t+dt]$ with probability $(dt/\tau)e^{-t/\tau}$, where $t$ is the time since the previous collision. The average time between collisions is hence

$$\langle t \rangle = \int_0^\infty t e^{-t/\tau} \frac{dt}{\tau} = \tau. \quad (8)$$

e) The time $T$ between the last and next collision average over all electrons is $2\tau$. That the time is larger than $\tau$ can be understood as follows: When we average over the electrons we choose a specific moment. When we do this there will be more electrons that are in a long time interval (between collisions) than in a short one.

To derive probability distribution for $T$ we choose a random electron. The fraction of electrons which will have a collision during the following time interval $[t_1, t_1+dt_1]$ is given by $e^{-t_1/\tau} dt_1/\tau$. Out of these a fraction $e^{-t_2/\tau} dt_2/\tau$ collided in the preceding time interval $[-(t_2+dt_2), t_2]$. Hence, the fraction of electrons which collided in $[-(t_2+dt_2), t_2]$ and will collide in $[t_1, t_1+dt_1]$ is hence

$$\frac{e^{-t_1/\tau} e^{-t_2/\tau} dt_1 dt_2}{\tau^2}. \quad (9)$$

Now we are interested in the time $T = t_1 + t_2$ between the collisions. Writing the previous result in terms of this we get

$$\frac{e^{-t_1/\tau} e^{-(T-t_1)/\tau} dt_1 dt T}{\tau^2} = e^{-T/\tau} dt_1 dT. \quad (10)$$

We now want to find the average value for $T = t_1 + t_2$. This is obtained as

$$\langle t_1 + t_2 \rangle = \int_0^\infty \int_0^\infty (t_1 + t_2) e^{-(t_1+t_2)/\tau} \frac{dt_1 dt_2}{\tau^2} = 2\tau. \quad (11)$$

### Problem 1.2 - Joule Heating

For thermal averages we have the following properties

$$\langle v \rangle = \langle \dot{v} \rangle = 0 \quad (12)$$

$$\langle \frac{mv^2}{2} \rangle = \langle T \rangle, \quad (13)$$

where $v$ is the velocity directly after a collision. The first condition says that direction is completely random after a collision. The second says that the speed is dependent of the temperature (at the point of the collision).

a) The force from the electric field is $-eE$ and we hence have $ma = -eE$ which gives the velocity at a time $t$ after a collision

$$v = v_0 - \frac{eE t}{m}. \quad (14)$$

After a time $t$ a second collision occurs after which the velocity is $v_1$. We then get the change in energy

$$\Delta E = \frac{mv_2^2}{2} - \frac{mv_0^2}{2} = \frac{mv_2^2}{2} = e\dot{v_0} \cdot E t + \frac{m}{2} \left( \frac{eE t}{m} \right)^2 - \frac{mv_1^2}{2}. \quad (15)$$
Now we take the thermal average over this. From (13) we get that \( \langle v_1^2 \rangle = \langle v_0^2 \rangle \) and also that the second term on the right hand side vanishes. Hence we get

\[
\langle \Delta E \rangle = \frac{(eEt)^2}{2m}.
\] (16)

**b)** The average loss per electron and per collision is now (using that the probability of collision at the interval \( t \) to \( t + dt \) after the previous collision is \( dt e^{-t/\tau} / \tau \)) is given by

\[
\int_0^\infty dt \frac{e^{-t/\tau} (eEt)^2}{2m} = \frac{eE^2 t}{m}.
\] (17)

If \( n \) is the number of electrons per cubic centimeter we get that the total loss per cubic centimeter and second is given by

\[
nc^2\tau^2E^2 = \frac{nc^2\tau E^2}{m} = \sigma E^2
\] (18)

where \( 1/\tau \) is the number of collisions per second. In a wire of length \( L \) and cross section \( A \) the volume is \( AL \) and we have the loss per second in the wire

\[
P = AL\sigma E^2 = \frac{A^2L\sigma E^2}{A\sigma} = \frac{A^2LJ^2}{A\sigma} = I^2 \frac{L}{A\sigma} = I^2 R.
\] (19)

**Problem 1.3 - Thomson Effect**

We now assume that the speed of the electron after a collision \( v_i = |v_i| \) is given by

\[
\frac{mv_i^2}{2} = E(T(r_i)),
\] (20)

where \( T(r_i) \) is the temperature at point \( r_i \) where the collision occurs (to simplify the problem we assume that the speeds are not randomly distributed but only the direction, see not in the end). We still assume that

\[
\langle v_i \rangle = 0,
\] (21)

because the direction is random after a collision. Again we have that for a single electron which collides at one point \( r_0 \) and then again at \( r_1 \) at a time \( t \) later is

\[
\Delta E = \frac{mv_0^2}{2} - ev_0 \cdot Et + \frac{m}{2} \left( \frac{eEt}{m} \right)^2 - \frac{mv_1^2}{2} = E(r_0) - E(r_1) - ev_0 \cdot Et + \frac{m}{2} \left( \frac{eEt}{m} \right)^2.
\] (22)

Now from regular mechanics we have

\[
r_1 = r_0 + \delta r = r_0 + v_0 t - \frac{eEt^2}{2m}.
\] (23)

Assuming now that the distance \( \delta r \) is small compared to the typical change in thermal energy we may expand \( E(r_1) \) around \( r_0 \) to get

\[
E(r_1) = E(r_0) + \delta r \cdot \nabla E(r_0) = E(r_0) + (\delta r \cdot \nabla T) \frac{dE}{dT},
\] (24)

where the last equation is obtained since the thermal energy depends on the position only through the temperature. Inserting this into Eq. (??) we get

\[
\Delta E = - \left( v_0 t - \frac{eEt^2}{2m} \right) \cdot \nabla T \frac{dE}{dT} - ev_0 \cdot Et + \frac{m}{2} \left( \frac{eEt}{m} \right)^2.
\] (25)
Now we can take the thermal average and the two terms which are linear in \( v_0 \) are then vanishes as all directions are equally probable. The result is then

\[
\langle \Delta E \rangle = \frac{eE t^2}{2m} \cdot \nabla T \frac{dE}{dT} + \frac{(eE t)^2}{2m}.
\]

Finally we average over time distribution of the collisions to get

\[
\int_0^\infty dt \frac{e^{-t/\tau}}{\tau} \left[ \frac{eE t^2}{2m} \cdot \nabla T \frac{dE}{dT} + \frac{(eE t)^2}{2m} \right] = \frac{eE x^2}{m} \cdot \nabla T \frac{dE}{dT} + \frac{(eE t)^2}{m}.
\]

Finally, the there is one collision per \( 1/\tau \) and the density of electrons is \( n \) so the energy loss per second and volume is

\[
P = \frac{net}{m} \frac{dE}{dT} \mathbf{E} \cdot \nabla T + \frac{n(eE)^2}{m} \tau,
\]

where the first part is due to the thermal gradient.

**Note on averages:** In the solution above we were a bit sloppy and assumed that the velocity \( v_i \) directly after a collision satisfies Eqs. (20) and (21). However, according to Drude’s one really should assume that not only the direction is random but also energies (and hence also the \(|v_i|\)) are random, i.e.

\[
\langle \frac{mv_i^2}{2} \rangle = \mathcal{E}(T(r_i))
\]

\[
\langle v_i \rangle = 0.
\]

We we do this we run into a bit of problem since we need to take the average to obtain the last step of Eq. (22). In order to solve the problem now, we have to separate two types of averages, one over the direction and one over the magnitude of the velocity. If we denote these \( \langle \cdot \rangle_{\text{dir.}} \) and \( \langle \cdot \rangle_{\text{mag.}} \) we then have

\[
\langle v_i \rangle_{\text{dir.}} = 0 \quad \langle \frac{mv_i^2}{2} \rangle_{\text{mag.}} = \mathcal{E}(T(r_i)).
\]

If we average \( v_i \) over the magnitude we get \( \bar{v}_v \mathbf{T} \) where \( \mathbf{T} \) is the thermal velocity and \( \bar{v} \) is a random unit vector which vanishes when we take the average over direction. Now, when solving the problem above we first take the average over magnitude to obtain the last step in Eq. (22). The average over the direction in order to get from Eq. (25) to (26).

### 1.4 - Helicon Waves

We have a magnetic field \( \mathbf{H} = H\hat{z} \) and an electric field \( \mathbf{E}(t) = \text{Re} \mathbf{E}(\omega)e^{-i\omega t} \) perpendicular to this, i.e. \( \mathbf{E}(\omega) = (E_x, E_y, 0) \). We want to consider circularly polarized electric fields so that \( \mathbf{E}(\omega) = (E_x, \pm iE_x, 0) \).

a) The momentum equation is then given by

\[
\frac{d\mathbf{p}}{dt} = -\frac{\mathbf{p}}{\tau} - e \left[ \mathbf{E} + \frac{1}{mc} \mathbf{p} \times \mathbf{H} \right].
\]

Assuming a function \( \mathbf{p}(t) = \text{Re} \mathbf{p}(\omega)e^{-i\omega t} \). Inserting this into the equation we get

\[
-i\omega \mathbf{p} = -\frac{\mathbf{p}}{\tau} - e\hat{x}E_x \mp ie\hat{y}E_y = \frac{eH}{mc}(p_y\hat{x} - p_x\hat{y})
\]

\[
-\frac{\mathbf{p}}{\tau} = \frac{eH}{mc}(p_y\hat{x} - p_x\hat{y}).
\]
If we divide this into components we get the equations

\[ \begin{align*}
-i\omega p_x &= -\frac{1}{\tau} p_x - e E_x - \omega_c p_y \\
-i\omega p_y &= -\frac{1}{\tau} p_y + i e E_x + \omega_c p_x \\
-i\omega p_z &= -\frac{1}{\tau} p_z.
\end{align*} \tag{35} \]

The last equation gives \( p_z = 0 \). The two other we write in matrix form

\[
\begin{pmatrix}
-i\omega + \frac{1}{\tau} & \omega_c \\
-\omega_c & -i\omega + \frac{1}{\tau}
\end{pmatrix}
\begin{pmatrix}
p_x \\
p_y
\end{pmatrix}
= -e E_x \begin{pmatrix} 1 \\ \pm i \end{pmatrix} \tag{36}
\]

Using the inverse of a two-by-two matrix (switch places on diagonal, switch sign on the other two and divide by the determinant) we get

\[
\begin{pmatrix}
p_x \\
p_y
\end{pmatrix}
= -\frac{e E_x}{(-i\omega + 1/\tau)^2 + \omega_c^2} \begin{pmatrix}
-i\omega + \frac{1}{\tau} & -\omega_c \\
\omega_c & -i\omega + \frac{1}{\tau}
\end{pmatrix}
\begin{pmatrix} 1 \\ \pm i \end{pmatrix} \tag{37}
\]

From this we get

\[
\begin{align*}
p_x &= -\frac{e E_x}{(-i\omega + 1/\tau)^2 + \omega_c^2} \left( -i\omega + \frac{1}{\tau} + i\omega_c \right) \\
p_y &= -\frac{e E_x}{(-i\omega + 1/\tau)^2 + \omega_c^2} \left( \omega_c \pm i \left( -i\omega + \frac{1}{\tau} \right) \right) = \pm ip_x.
\end{align*} \tag{38}
\]

Now we have the relation

\[ j = -\frac{ne}{m} \mathbf{p} \tag{39} \]

so we get

\[
\begin{align*}
j_x &= \frac{ne^2 E_x}{m} \left( -i\omega + \frac{1}{\tau} \mp i\omega_c \right) = \frac{ne^2 E_x}{m} \left( \frac{1}{\tau} - i\omega \mp i\omega_c \right) \\
&= \frac{ne^2 E_x}{m} \left[ \frac{1}{\tau} - i(\omega \pm \omega_c) \right] \\
&= \frac{ne^2 \tau E_x}{m} \frac{1}{1 - i(\omega \mp \omega_c)\tau} \\
&= \frac{\sigma_0}{1 - i(\omega \mp \omega_c)\tau} \frac{E_x}{\tau}. \tag{40}
\end{align*}
\]

Note that this is a correction to the conductivity tensor \( \sigma_0 \) which is now given by the term above, i.e.

\[ \sigma = \frac{\sigma_0}{1 - i(\omega \mp \omega_c)\tau} \tag{41} \]

b) Looking for a solution of the form \( \mathbf{E} = E_0(1, \pm i, 0)e^{ikx - i\omega t} \) we insert this into Eq. (1.34) in the book

\[ -\nabla^2 \mathbf{E} = \frac{\omega^2}{c^2} \epsilon(\omega) \mathbf{E}, \tag{42} \]

we get that \( k \) and \( \omega \) must satisfy the equation

\[ k^2 \epsilon' = \omega^2 \epsilon(\omega). \tag{43} \]

Furthermore we have (from the book) that

\[ \epsilon(\omega) = 1 + \frac{4\pi i \sigma}{\omega}. \tag{44} \]
Inserting our conductivity tensor we have
\[
\epsilon(\omega) = 1 + \frac{4\pi i}{\omega} \frac{\sigma_0}{1 - i(\omega + \omega_c)\tau} = 1 + \frac{4\pi i}{\omega} \frac{ne^2\tau}{m(1 - i(\omega + \omega_c)\tau)} = 1 - \frac{\omega_p^2}{\omega} \frac{1}{\omega + \omega_c + i/\tau},
\]
where \(\omega_p^2 = 4\pi ne^2/m\) is the plasma frequency (SI-units \(\omega_p^2 = ne^2/(\epsilon_0 m)\)).

c) Now we wish to solve the equation to see what the solutions are. Choosing \(E_y = +iE_z\) (the upper sign) we have the equation
\[
k^2c^2 = \omega^2 - \frac{\omega_p^2}{\omega - \omega_c + i/\tau}
\]
We now need some relevant values. One condition that is wanted is that \(\epsilon H/m = \omega_c\tau \gg 1\). This implies that the magnetic field should be strong and that the collision time is not too short. In the book they have \(\tau \sim 10^{-14}\) s which is much too short since it would require a field of a few 100 T.

However, by lowering the temperature to a really low value the collision time can be increased significantly. In potassium (K) at 4.2 K the collision time turns out to be around \(10^{-10}\) s (we have used \(n \sim 10^{22}/\text{cm}^3\) and that the resistivity is \(\rho = 1/\sigma = 2.0 \times 10^{-13}\) \(\Omega m\)). If we further assume that we have a magnetic field of around 10 T we get \(\omega_c\tau \sim 100\) so that our condition is fulfilled.

In CGS-units we use (SI-units within parenthesis)
\[
n = 10^{22}/\text{cm}^3 \quad (10^{28}/\text{m}^3)
\]
\[
c = 3.00 \times 10^{10}\text{ cm/s} \quad (3.00 \times 10^8\text{ m/s})
\]
\[
e = 4.80 \times 10^{-10}\text{ esu} \quad (1.60 \times 10^{-19}\text{ C})
\]
\[
m = 9.11 \times 10^{-28}\text{ g} \quad (9.11 \times 10^{-34}\text{ kg})
\]
\[
H = 10^5\text{ gauss} \quad (10\text{T})
\]

For these typical values we have
\[
\omega_p = 5.6 \times 10^{15}/\text{s} \approx 10^{16}/\text{s}
\]
\[
\omega_c = 1.8 \times 10^{12}/\text{s} \approx 10^{12}/\text{s}
\]
\[
\tau = 10^{-10}\text{ s}
\]

From Eq. (46) we note that a real \(k\) (imaginary \(k\) implies damping) is obtained if \(\epsilon(\omega) > 0\). First we consider \(\omega > \omega_p\). Using that \(\omega_p \gg \omega_c\) and that \(\omega\tau > \omega_p\tau \gg 1\) we have
\[
\epsilon(\omega) = 1 - \frac{\omega_p^2}{\omega} \frac{1}{\omega - \omega_c + i/\tau} = 1 - \frac{\omega_p^2}{\omega} \frac{\tau}{\omega\tau - \omega_c\tau + i} \approx 1 - \frac{\omega_p^2}{\omega^2}
\]
and we see that this is indeed larger than unity. For \(\omega < \omega_c\) we have
\[
\epsilon(\omega) = 1 - \frac{\omega_p^2}{\omega} \frac{1}{\omega - \omega_c + i/\tau} \approx 1 + \frac{\omega_p^2}{\omega} \frac{1}{\omega - \omega_c} > 1.
\]

d) For \(\omega \ll \omega_c\) we have
\[
\epsilon(\omega) \approx \omega_p^2 \omega_c
\]
which yields
\[
k^2c^2 = \omega^2 \omega_p^2/\omega_c^2 \quad \text{or} \quad \omega = \omega_c \left( \frac{k^2c^2}{\omega_p^2} \right).
\]

For our values together with a wavelength of \(\lambda = 1\ \text{cm}\) \((k = 2\pi/\lambda)\) this gives
\[
\omega \sim 0.1/\text{s}.
\]
For 10 kgauss = 0.1T we get \(\omega \sim 10^{-3}/\text{s}\). Really slow oscillations!