Larry Spruch and the Three-Body Problem

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ABSTRACT

Larry Spruch has made important contributions to both the three-body bound state and scattering problems. I review these contributions and bring some topics up to date. I have embedded the most important references in the paper, but a short selected bibliography is appended.

INTRODUCTION

I am honored by the request for an article reviewing the contributions of Larry Spruch to the three-body problem. I am proud to call Larry, a "physicist's physicist," both mentor and friend. I have learned much from him over many years, and writing this review has in itself been very educational.

The problem of three interacting particles goes back to the earliest days of quantum mechanics. Three-body systems abound. In molecular physics we have the H$_2^+$ molecule, in atomic physics the helium atom (and its isoelectronic ionic relatives), in nuclear physics the H$^3$ (triton) and He$^3$ (helion) nuclei, and in particle physics the three-quark baryons. As is taught in textbooks, the two body problem can be reduced by a center-of-mass transformation to an effective one-body problem, thereby reducing the number of degrees of freedom from six to three (we are ignoring internal degrees of freedom, such as spin). Unfortunately the same transformation in the case of three bodies only reduces the degrees of freedom from nine to six. While the hydrogen atom, the prototypical two-body system, yielded immediately to analysis even in the old quantum theory, there was no such luck for the helium atom -- one of the simplest and most important three body systems. The helium atom has been called "an atom with one electron too many." So, with the birth of the new quantum theory in 1926, the helium atom became the target of theorists. This problem cannot be solved in closed form but, armed with perturbation theory and Rayleigh-Ritz variational methods, Unsold, Heisenberg, Hylleraas, Hartree, and Fock, soon achieved good agreement with experimental values for the ground state energy. Finding ever more precise values for helium energy levels (and others in the helium isoelectronic sequence) has become a kind of competitive industry, which is reviewed by Gordon Drake in his own contribution to the AMO Handbook [Atomic, Molecular & Optical Physics Handbook, Ed. By Gordon W.F. Drake, AIP Press, N.Y., 1995].

Slightly before the birth of the new quantum mechanics, in 1923, Henderson discovered the process of charge capture when alpha particles, passing through mica, sometimes emerged as neutral helium atoms or He$^+$ ions, and the need for analysis of the three-body scattering problem became apparent. (Capture is kinematically forbidden for two elementary bodies; a third body is needed to absorb some of the momentum.) In 1927 L.H. Thomas published a semi-classical analysis of the capture problem – a
significant paper to which we will return. The first purely quantum mechanical calculations were done by Oppenheimer in 1928 and by Brinkmann and Kramers in 1930.

Larry Spruch's interest in the three-body problem actually began with his Ph.D. dissertation. Under the supervision of Leonard Schiff, Larry analyzed the beta decay of the triton. Since I have not seen this work, I cannot comment on it further. The majority of his research has been on the atomic, rather than the nuclear problem.

Research on the two parts of the spectrum of the three-body atomic hamiltonian have led largely separate existences. Of course, the bound states and the continuum are connected by a common hamiltonian, and theorems such as Levinson’s demonstrate this clearly. Larry has often emphasized the importance of weakly bound states to low-energy scattering. His variational upper bound on the scattering length requires a precise knowledge of the number of bound states of the projectile-target atom system. Larry’s important contributions to these topics which bridge the two parts of the spectrum are discussed in the contribution by Yukap Hahn and by Larry himself elsewhere in this collection. In this review I shall treat the bound state problem first and then turn to the three-body scattering problem.

BOUND STATES

For Larry, the major interest in the three-body bound state problem has been in the existence or non-existence of bound states of various combinations of three charged particles (in some cases, he has considered four particles, and also nuclear rather than pure Coulomb potentials). The hamiltonian is, in an obvious notation,

\[ H = T + V = \sum_{i=1}^{3} \frac{p_i^2}{2m_i} + \sum_{i<j} \frac{q_i q_j}{r_{ij}} \]

This hamiltonian commutes with the total momentum operator \( P = \sum p_i \). It is thus useful to reduce the degrees of freedom from nine to six by transferring to center of mass coordinates, in which the variable conjugate to \( P \), i.e. \( R = \sum m_i r_i / \sum m_i \) is constant (usually zero). This can be done in a variety of ways, and the choice depends on the particular problem. The use of Jacobi coordinates leaves the kinetic energy term in a diagonal form but complicates the potential function. Other methods leave the potential relatively simple but introduce a Hughes-Eckart “mass polarization” cross-term in the kinetic energy. We will not dwell on the details here, but assume that the center-of-mass motion has been removed, and will use the notations \( H, T \) and \( V \) for the hamiltonian, kinetic energy and potential energy operators in the center of mass coordinates.

For any bound states to be possible clearly at least one of the charges \( q_i \) must differ in sign from the other two. Bound states are defined by negative eigenvalues in the point spectrum of \( H \). Some of these, such as the doubly-excited states of helium embedded in the continuum, are autoionizing and hence unstable. We shall primarily be interested in stable states, i.e. eigenvalues below the continuous spectrum.

Because of scaling properties of the hamiltonian, the existence and stability of bound states really depends on two mass ratios and two charge ratios. These parameters
define a four-dimensional parameter space. There exist three-dimensional hypersurfaces in this space which separate regions where there are no bound states from those where bound states exist. In some circumstances it is sensible to take one of the masses to be infinite, reducing the parameter space to three dimensions. This can be used to eliminate the mass polarization term and represents a significant simplification, but it clearly eliminates the study of some interesting systems, such as the positronium negative ion first posited by Wheeler in 1946 and experimentally found by Mills thirty-five years later.

Considerable work has been done especially in the special case of three charges of equal magnitude, which includes the negative hydrogen ion $H^- = (pe^-e^-)$, the hydrogen molecular ion $H_2^+ = (ppe^-)$, the aforementioned positronium ion $Ps^- = (e^+e^-e^-)$, as well as exotic versions of these, in which negative muons (or even antiprotons) are substituted for electrons, and deuterons or tritons for protons, etc. These latter systems are of particular interest in the studies of muon-catalyzed fusion.

In testing for the existence of a bound state, it is sufficient that a variational upper bound, such as a Rayleigh-Ritz estimate, give a negative energy. That is, if a normalized trial function $\psi_i$ can be found such that $\langle \psi_i | H | \psi_i \rangle < 0$, then a bound state must exist. Of course, to be a stable state the energy of this state must be less than the energy of any separated subsystem. Once the existence of one bound state is demonstrated, the question then arises as to how many bound states the system may have, and are they stable? A lower limit on the number of bound states, and upper limits on their energies, can be found by the Hylleraas-Undheim theorem. But these methods are powerless to prove the non-existence of bound states. This requires that a lower bound on the energy have a non-negative value. There is no room for error – a slightly negative lower bound leaves room for a bound state.

Larry’s first published paper on the three-body problem, co-authored by Martin Kelly, deals with the classic problem of the ground state energy of the helium atom and its isoelectronic sequence [Phys. Rev. 116, 911-913 (1959)]. This work uses perhaps the simplest wave function, having a single variational parameter, which suffices to prove the existence of $H^-$, the negative hydrogen ion. Because of its very weak binding (0.75 eV) this simple two-electron ion has long been a rigorous test of theoretical techniques. It is, of course, much more than a theorist’s playground. Besides its important role in understanding the sun’s opacity, intense beams of $H^-$ ions are now being produced and studied as possible injectors for tokamak fusion reactors.

Spruch and Kelly modified a wave function of Pluvinage, used it in a Rayleigh-Ritz calculation with the charge parameter in the wave function as the only variational parameter, and found $E < -0.5005$ a.u. Since this is less than the $-0.5000$ a.u. ground state energy of the hydrogen atom, it proves that the $H^-$ ion exists and is stable. Although the true value of the energy is $-0.5277$ a.u., it is nevertheless quite an accomplishment to show binding with a one-parameter simple wave function. Spruch and Kelly also generalized their technique to three particles of arbitrary mass, and applied it to the $pp\mu^-$ system, but the results were poor.
The hydrogen negative ion $H^-$ is particularly interesting mathematically because it has been proven rigorously to have only the single stable bound state [R.N. Hill, Phys. Rev. Lett. 38, 643 (1977) and J. Math. Phys. 18, 2316 (1977)] whereas all other two-electron species have infinitely many. Thirring [A Course in Mathematical Physics, Vol. 3 (Wien-New York, 1979)] has shown that if one starts with the helium atom, and continuously reduces the nuclear charge, the bound states are pushed into the continuum, leaving the one state when $Z = 1$. Drake [Phys. Rev. Lett. 24, 126 (1970); see also J. Midtal, Phys. Rev. 112, A1010 (1965)] has shown that the $2p^2\,^3P$ state is also extremely weakly bound, lying just 0.0095 eV below the $n = 2$ continuum (but above the $n = 1$ continuum - parity inhibits its decay). This state has not been seen experimentally. For the even tougher problem of the positronium ion, only the ground state is known to be stable.

Subsequent work by Larry and his coworkers on three-body bound states consists of five papers, published from 1968 to 1990. Larry’s first in-depth look at the question of proving the existence or nonexistence of bound states came during his sabbatical in Oxford in 1964. At a conference in Yugoslavia on the few-nucleon problem [Proceedings of the Ninth Summer Meeting of Nuclear Physicists, (Hercegnovi, Yugoslavia, 1964), pages 271-317] Larry reviewed several possibly useful necessary conditions for bound states to exist, primarily for a single particle in an external potential (or equivalently, for two particles with only an internal potential). Multiple particle systems are discussed only briefly: an integral equation – Green’s function approach, generalizing a method for the one-particle potential via hyperspherical coordinates, and a Feshbach projection operator approach, but these apparently were not subsequently implemented. Nevertheless, these ideas evolved into an adiabatic formalism that is at the core of the five papers, the first of which, Gertler, et al (see below) was submitted in January 1968. The ideas are updated and briefly summarized in Larry’s 1968 Boulder Lectures [Lectures in Theoretical Physics, edited by S. Geltman, K.T. Mahanthappa, and W.E. Brittin, vol. XI-C (New York, 1969), pages 57-75]. The adiabatic approach is reminiscent of the Born-Oppenheimer method but is not restricted to large masses. The method is physically appealing and is capable of being made mathematically rigorous, although except for one special case, I am not aware of this having been done.

The first published paper using the adiabatic approach is with Fred Gertler and Herschel Snodgrass [F. H. Gertler, H. B. Snodgrass, and Larry Spruch, Phys. Rev. 172, 110 (1968)]. Briefly, it deals with the $n$-body bound state problem by reducing it to a one body moving in an effective potential defined by the others, which are “frozen” in position. The adiabatic potentials are known from other workers. The idea is an outgrowth of work by Hahn and Spruch in the scattering problem, which is apparently unpublished. The key to the method is the necessary condition for the existence of a bound state for a particle in a potential $V(r)$: if we assume that $V$ vanishes at infinity, and if we define $V_-(r) = \min(V(r),0)$ then the necessary condition is

$$(2m/\hbar^2) \int_0^Z (-V_-(r)) dr \geq 1,$$

according to a theorem due to Jost, Pais, Bargmann, and Schwinger (which I will call the JPBS theorem). (More generally, if we are considering bound states with angular momentum $L$, the “one” on the right side of the inequality can be replaced by $(2L+1)N_L$, where $N_L$ is the number of bound states of angular momentum $L$. We are primarily interested in $L = 0$, and $N_L = 1$.) Thus, if an adiabatic potential can be
found which is expected to be a good description of the motion of one particle in the field of the remaining fixed particles, and if the integral turns out to be substantially less than one, there is good reason to doubt the existence of a bound state. Consider now the system of the helium atom $\text{He} = (\alpha e^- e^-)$. Gertler, et al. show that a positron cannot bind to this to form a four-body composite system. Indeed, if one replaces the positron by a particle of the same charge but mass $m^+$ and fixes the positions of this particle and the alpha particle, one can find an adiabatic potential in the literature which, inserted into the left side of the JPBS theorem, results in a value of $0.421 m^+/m_e$. The theorem shows that a bound state cannot form unless $m^+$ exceeds $2.38 m_e$, which excludes the positron. (Results of this type help to map out the hypersurface separating the bound from the non-bound regions of the four dimensional parameter space mentioned earlier in this section.) They similarly prove that no particle of unit positive charge and arbitrary mass can bind to the $\text{He}^+$ ion - not a positron, proton, etc. They fail, however, to resolve the question of whether a positron can bind to a neutral hydrogen atom. The adiabatic potential would allow a bound state, and the method only gives a negative lower bound on its energy. This system is the subject of the next paper.

Three years later, Larry together with Ira Aronson and Chemia Kleinman improved the previous procedure and were able to demonstrate the nonexistence of the positron-hydrogen three-body bound state $(\text{pe}^- \text{e}^+)$ [Phys. Rev. A 4, 841 (1971)]. They achieved this by fixing the distance $r$ between the proton and the positron, but not the orientation of the vector $\vec{r}$. They then work in the subspace of zero total angular momentum. They indicate that no bound state exists if the positron is replaced by a particle of unit positive charge and mass less than $1.48 m_e$. The proof of the nonexistence of the bound state is not rigorous because no error bounds on the adiabatic potential are given. A rigorous proof was eventually given by E.A.G. Armour [J. Phys. B 11, 2803 (1978)] who found a rigorous lower bound to the adiabatic potential using the method of Temple and Kato.

In 1975, Aronson, Kleinman and Spruch attempted to overcome the difficulty in proving the nonexistence of bound states arising from the lack of a margin of error. They provide such a margin by replacing the true hamiltonian by $H(\lambda) = H_{\text{true}} + \lambda \nu$ where $\nu$ is a weak attractive potential and $\lambda$ is a positive coupling constant. Employing a Rayleigh-Ritz calculation, the estimate the ground state energy $E(\lambda)$. They study the behavior of $E(\lambda)$ in the vicinity of the threshold at which $H(\lambda)$ can just support a bound state. If this value is not too small then it is unlikely that $H_{\text{true}}$ can support a bound state.

Larry made one last return to this fascinating problem in 1990, with a pair of papers with his post-doc Zonghua Chen [Chen and Spruch, Phys. Rev. A 42, 127 & 133 (1990)]. The first paper revisits the question of lower bounds on the ground state energy and conditions necessary for the existence of a bound state in the case of a single particle in an external potential $V$, in both one and three dimensions. In the latter case $V$ need not be spherically symmetric. They go back to the Green’s function techniques discussed in Larry’s 1964 Hercegovni and 1968 Boulder lectures, reprise some of the material found there, and then derive somewhat stronger bounds and conditions. These are then applied in the second paper to the cases of diatomic molecular ions. They show that one electron cannot bind two nuclei if the sum of the charges of the nuclei is greater than or equal to
three. They also prove that the four-particle system consisting of two nuclei with equal charges greater than or equal to three and two electrons is also unbound. These results have recently been confirmed in a more rigorous analysis by Krikeb, et al. [A. Krikeb, A. Martin, J.-M. Richard, and T. T. Wu, Few-Body Systems, 29, 237 (2000)].

THREE-BODY SCATTERING

The total corpus of Larry’s contributions to scattering is so large, that I have confined my discussion to those works which directly bear on the three body problem, and in particular, those which have most directly influenced me. In addition, this section gives me the opportunity for some personal reflection; I was not personally involved with the bound-state research.

When I arrived to begin post-doc research with Larry in September 1967 I was remarkably ignorant of atomic physics, scattering theory, etc. My background as a student of Julian Schwinger was in quantum field theory, especially quantum electrodynamics. I did have a grounding in the concept of variational principles. Shortly after I had settled in, Larry showed me a Physical Review Letter by John Nuttall [Phys. Rev. Lett. 19, 473 (1967)]. The paper was entitled “Kohn Variational Principle for Three-Particle Scattering” and the abstract read: “The Kohn variational principle is extended to apply to scattering processes where a two-particle bound state is broken up by a third particle.” Larry said that he was under the impression that Lenny Rosenberg had derived Kohn variational principles for the general case of \( m \) particles in the initial state being transformed into \( n \) particles in the final state (Nuttall’s claim was that he had deduced the \( m=2 \) to \( n=3 \) Kohn VP). As I researched this problem, two things became clear. First, Nuttall’s claim in the abstract was misleading or incorrect. While he had included the three-body breakup channel in his trial wave function, the breakup amplitude was not treated variationally, only the elastic amplitude was. That is, the error in the breakup amplitude was first order, rather than second order. Nuttall’s paper, however, nevertheless contained much interesting material. It should be emphasized that Nuttall’s work, and ours, was restricted to short-range potentials.

The second point was that, while Lenny had indeed published a paper giving the general Kohn variational principle for \( m \)-to-\( n \) scattering [Phys. Rev. 134, B937 (1964)], his derivation relied upon a very naïve asymptotic form of the wave function, a point which was clear from Nuttall’s paper. Rosenberg had assumed that, by analogy with the two-body case, the wave function asymptotically contained just a plane wave part, representing the incident wave, and a scattered part consisting of a product of bound state wavefunctions for those subsystems that remain bound multiplied by a multidimensional spherical wave representing the escaping components. The requisite scattering amplitude was the coefficient of the latter term. Nuttall’s wave function contained additional terms, showing that Rosenberg’s assumed form was not valid in all regions of configuration space. In the 2-to-3 case, for example, there are terms where two particles are not bound but separating with very little relative velocity. These regions of configuration space have measure zero, but infinite volume, and in these regions the fall-off of the wave function is slower than that of the spherical wave. This can be understood in terms of multiple scattering of the particles before separation. Therefore, the contribution to the Kohn
variational integral is of the form zero times infinity, and it is not clear how significant is this contribution.

At this point Larry decided to bring Lenny into the discussion. We agreed that a new derivation was needed. At this distance in time I cannot explain why the final papers were not finished and submitted until 1971, when I returned to NYU for a summer. Nor do I recall why we chose to submit these papers to the “D” section of Physical Review, which has led to their being overlooked several times since as repeated overlapping treatments have appeared in the “C” and “A” sections. [I believe that the division of Physical Review into sections by subject has been a disaster for the unity of physics as a discipline; only Physical Review Letters is read by all physicists.] In any event we submitted the first paper in July and the second in October. They were published back-to-back [M. Lieber, Leonard Rosenberg and Larry Spruch, Phys. Rev. D 5, 1330 & 1347 (1972)].

The Kohn VP follows from an identity established by Kato. For simplicity, here I restrict the discussion to the case of potential scattering, and refer to the papers for the details. Kato’s identity can be written

\[ T_{fi(\text{exact})} = T_{fi(\text{trial})} + (\Psi_{f(\text{exact})}^{(-)} , (H - E) \Psi_{i(\text{trial})}^{(+)} ) . \]

Here \( T_{fi} \) is the scattering amplitude for the process \( i \to f \). On the left side of the identity is the exact amplitude and on the right side is an approximate amplitude contained in the asymptotic (outgoing spherical wave) portion of the approximate wave function \( \Psi_{i(\text{trial})}^{(+)} \). This wave function evolves from a plane-wave state representing the incident particle. The other wave function, \( \Psi_{f(\text{exact})}^{(-)} \) is the exact solution the Schrödinger equation which evolves from a plane wave with the final state momentum and with an incoming spherical wave boundary condition (time-reversed solution). We note that if we substitute into the Kato identity the exact incident wave function in place of the trial wave function, the matrix element term vanishes and we have the tautologous relation \( T_{fi(\text{exact})} = T_{fi(\text{exact})} \). On the other hand, if we insert just the incident plane wave part as a trial function, \( \Psi_{i(\text{trial})}^{(+)} \to \Phi_i \) which therefore has no scattering amplitude term, we deduce the familiar formula \( T_{fi(\text{exact})} \approx (\Psi_{f(\text{exact})}^{(-)} , \Phi_i ) \). The Kohn VP results from the Kato identity when one replaces the exact wave function \( \Psi_{f(\text{exact})}^{(-)} \) with an approximate trial wave function satisfying the same time-reversed boundary conditions. If we substitute their plane wave parts for both wave functions, we get \( T_{fi} \approx (\Phi_f , \Phi_i ) \), i.e. the first Born approximation.

In applying the same ideas to the 2-to-3 breakup amplitude the \( \Psi_i^{(+)} \) represents the three-body state that evolves from the incident plane wave of the projectile and the bound pair. This is the wave function discussed by Nuttall. But we also need the wave function \( \Psi_f^{(-)} \), which is the wave function that evolves from three free particles, and includes 3-to-2 and 3-to-3 processes. This wave function was not completely known in all regions of configuration space, only where all three particles are far apart, or where two are bound. We worked quite hard to solve this problem. I recall presenting a progress report at a
DEAP meeting (the APS division now known as DAMOP) in New York. Ed Gerjuoy was the moderator. After my talk he told me that he was also working on this problem, and asked if we had found the “logarithmic terms.” We had not, and his question caused me to panic and delayed our research. No logarithmic terms exist. A second difficulty encountered was that some of the integrals we needed in the derivation had oscillatory divergences, although the final VP does not have divergent integrals. This we “fudged” using a recognized mathematical technique: Cesàro summation.

In the second paper we gave a more rigorous derivation based upon the Faddeev equations, and also discussed the Schwinger VP and its relation to the Kohn VP. This too was rediscovered later by others in other sections of Physical Review [e.g. Takatsuka, et al., Phys. Rev. A 24, 1812 (1981)]. Almost simultaneous with the publication of our papers, Bryce and Mandl showed how we could have simplified our derivation by avoiding the Cesàro summation [J. Phys. B 5, 912 (1972)].

None of us ever pursued the actual application of the Kohn VP we had derived, nor generalized it to the \( m \) to \( n \) case. The latter remains an open project. However, I became very interested in the question of asymptotic behavior of wave functions, and have contributed some results for the case of three charged particles [M. Lieber and A.M. Mukhamedzhanov, Phys. Rev. A 54, 3078 (1996)]. The original pair of papers still does receive citations, e.g. M. Viviani, A. Kievsky, and S. Rosati, Few-Body Systems 30, 39 (2001).

Another aspect of three-body scattering to which Larry made major contributions and which had a major influence on me, was his work with Robin Shakeshaft on the capture problem. Robin has summarized this work in his contribution to this collection, so I can afford to be brief.

In my historical introduction above, I mentioned L.H. Thomas’ remarkable 1927 semiclassical calculation of the cross section for capture of an electron in ion-atom scattering. According to his result the cross section should decrease at large energies according to a power law: \( \sigma \sim v^{-11} \) or \( \sigma \sim E^{-5.5} \), where \( v \) and \( E \) are the velocity and energy of the projectile ion in the lab frame. However, when in the next year Oppenheimer and then in 1930 Brinkmann and Kramers did fully quantum mechanical calculations (they did a Born approximation calculation, ignoring the projectile-nucleus repulsion), they found instead that \( \sigma \sim v^{-12} \) or \( E^{-6} \). As data accumulated it also appeared that the OBK cross section was too large by an order of magnitude. The discrepancy in the energy law remained a mystery until the 1955 Ph.D. dissertation of R.M. Drisko [unpublished] who estimated the second Born term and showed that it decreased with energy more slowly than the first Born term of the OBK, by exactly one power of \( v \), so that at sufficiently high velocity the second Born term will dominate over the first Born. But what was the physics behind this?

Thomas considered protons colliding with hydrogen atoms in their ground state. He uses quantum mechanics to describe the latter, but the collision process was described completely classically. Classically, the capture of the electron cannot occur with a single collision; one cannot conserve both momentum and energy. Instead Thomas showed that
two collisions would suffice. In the first, the proton collides with the electron, bringing it up to the speed needed for capture, but in the wrong direction. The electron then collides elastically with the target nucleus, changing its direction to match the projectile proton so that it can be captured. The pair emerges at the “Thomas angle” of \( \theta_T = \sqrt{3} m_e / 2 M \), where \( M \) is the projectile mass. In the quantal calculation, the double collision corresponds to a second Born term. The first Born term (OBK), corresponding to a classically forbidden single collision does not vanish because of the uncertainty principle, but decreases more rapidly. The experimental differential cross section at very high energy indeed shows a peak emerging at precisely the Thomas angle, verifying the scenario.

The literature on this subject is vast – even the number of review articles is large. Probably no review however is more useful than that of Robin Shakeshaft and Larry Spruch [Rev. Mod. Phys. 51, 369 (1979)]. This review renders superfluous any attempt by me to summarize Larry’s many contributions, which go back to 1973. Instead I will just pick one article which appeared after this review, in 1984, written also with Robin.

In this article [Phys. Rev. A 29, 605 (1984)] Robin and Larry demonstrate that there is a second classical scenario which can lead to capture: the projectile knocks the electron into its final state. The projectile then collides with the target nucleus, which changes its direction to match that of the electron, which is then captured. In this case the final pair emerges at an angle of 60 degrees. Thus there should be a second peak in the differential cross section at 60 degrees. This has not been seen experimentally.

In 1987 I was visiting Jim McGuire at Kansas State University. We had both been working on the then popular but now defunct theory of electron capture called the Strong Potential Born approximation. While there I was led to look at classical capture mechanisms for arbitrary mass particles. I discovered that there was a third logical possibility, but that this possibility and the one described in the previous paragraph could not both occur – which one is allowed depends on the masses. I drew up a diagram showing which regions in the \( m_1/m_2 \) vs. \( m_3/m_2 \) plane. I later discovered that a similar (but more opaque) diagram was published by Dettmann [K. Dettmann, Springer Tracts in Modern Physics 58, 119 (1971)], and so never published my diagram or analysis. This diagram proved sufficiently useful to Jim and his coworkers that eventually the published the analysis. To my chagrin, the diagram has become known as the “Lieber diagram” and was published by Jim in his article in the AMO Handbook.

The diagram shows that for certain mass ratios, classical kinematics forbids all three double collision capture processes. The question then arises, is capture by a triple collision then possible? If so, what does this do to the cross section? Is there a “Thomas peak”? I started working on this problem in 1987, but put it aside until recently. I have been investigating this problem now with my student, John Carter. We have discovered interesting kinematical features in the “forbidden zone” of the Lieber diagram. But it appears that the cross section is, contrary to my hope, still dominated by the first and second Born terms which both fall off as \( E^{-6} \). We estimate that the third Born term falls off as \( E^{-7} \ln E \) if triple collision capture is allowed, and as \( E^{-7} \) if not.
SUPPLEMENTARY BIBLIOGRAPHY

In addition to the references cited in the text I wish to add a few more that may be useful.


I wish to cite also the following papers that relate to the existence and stability of bound states, and from which many other references may be traced:


Even the wave function for the helium atom is still the subject of research:

For scattering I might add the monographs:
Roger G. Newton, Scattering Theory of Waves and Particles, 2nd Ed. (Springer, NY, 1982). There is a discussion of the asymptotic three-body wave function (short range potentials only) in section 17.4.5. [Gerjuoy’s work on this problem is summarized in E. Gerjuoy, Phil. Trans. Royal Soc. London A 270, 197 (1971)].
