Nature vs. Nurture: Predictability in Zero-Temperature Ising Dynamics

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Consider a dynamical many-body system with a random initial state subsequently evolving through stochastic dynamics. What is the relative importance of the initial state (“nature”) vs. the realization of the stochastic dynamics (“nurture”) in predicting the final state? We examined this question for the two-dimensional Ising ferromagnet following an initial deep quench from \( T = \infty \) to \( T = 0 \). We performed Monte Carlo studies on the overlap between “identical twins” raised in independent dynamical environments, up to size \( L = 500 \). Our results suggest an overlap decaying with time as \( t^{-\theta_h} \) with \( \theta_h = 0.22 \pm 0.02 \). This “heritability exponent” may equal the persistence exponent for the 2D Ising ferromagnet, but the two differ more generally.

Introduction. Given a typical initial configuration of a thermodynamic system, which then evolves under a specified dynamics, how much can one predict about what the system will look like at later times? In this paper, we study this problem in the relatively simple setting of a homogeneous Ising ferromagnet on a square lattice following a deep quench from infinite to zero temperature; in particular, we are interested in the influence of the initial state on the configuration at a later time \( t \). More colloquially we are investigating the role of “nature” vs. “nurture” in Ising systems, “nature” representing the influence of the initial configuration and “nurture” representing the influence of the single-spin dynamics, which, even at zero temperature, retains an element of randomness in the order that spins are chosen to attempt to flip.

This nature vs. nurture problem can be solved exactly for 1D random ferromagnets and spin glasses [1]. An initial attempt at a numerical study of the same problem in the 2D homogeneous ferromagnet on a square lattice was reported in [2]; the results, while suggestive, were inconclusive. In this paper we carry the earlier studies to a much stronger conclusion, so that (numerically, at least) the problem in the uniform 2D case has been largely solved.

This version of the problem of determining the effect of initial conditions on subsequent configurations was first proposed in [3]. It is part of a larger set of dynamical problems that can roughly be phrased as whether and under what conditions a system in the thermodynamic limit settles down to an equilibrium state after a finite time [1, 3]. This is understood in the sense of local equilibration: choose a region of large fixed size surrounding the origin, and ask whether after a finite time (which of course will generally depend not only on the system studied, but also on the choice of initial condition, dynamics, lattice type and dimensionality, and possibly other factors) domain walls cease to sweep across the region.

Local equilibration has been shown to occur in numerous systems, for example, any random Ising spin system with couplings chosen from a continuous distribution [1], but has also been shown not to occur for others, such as the 2D homogeneous Ising ferromagnet, under consideration here [3]. This situation is called local nonequilibrium (LNE) [3]. (Recall that this discussion concerns the infinite-size system.)

When LNE occurs, the configuration in any fixed, finite region never settles down. But one can still ask the following question: if one averages over all dynamical realizations, does the dynamically averaged configuration have a limit? If so, this limit can be thought of as a (dynamically) mixed ground state, in the same sense as one talks about mixed thermodynamic states at positive temperature. Or does even this averaged configuration not settle down?

The first possibility (a limit of the dynamically averaged configuration) can be thought of as “weak LNE”, while the second (no limit) is referred to as chaotic time dependence (CTD) [3]. As shown in [3], weak LNE implies a complete lack of predictability (“nurture” — after some time, the dynamics wipes out all information about the initial state), while CTD implies that some amount of predictability remains (“nature”). Moreover, this amount should be quantifiable, as in the 1D case [1]. So the nature vs. nurture question can be reframed as
whether weak LNE or CTD occurs for a given system.

The above discussion illustrates that the study of nature vs. nurture provides a great deal of information on some central dynamical issues concerning different classes of models. Our work is also related to the general area of phase ordering kinetics [4]. Two lines of work are particularly relevant to our study: Krapivsky, Redner and collaborators [5–7] investigated both the 2D and 3D Ising models with zero temperature Glauber dynamics to understand the time scales and final states of the dynamics. Derrida, Bray and Godreche introduced the persistence exponent [8], which characterizes the power law decay of the fraction of spins that are unchanged from their initial value as a function of time after a quench. This exponent was measured for the zero temperature 2D Ising model by Stauffer [9] and calculated exactly for 1D by Derrida, Hakim and Pasquier [10]. We will examine the relationships between the work reported here with these earlier results.

Heritability. To investigate nature vs. nurture for the 2D Ising ferromagnet we carry out a “twin” study. We start two Ising systems with the same infinite temperature initial condition and then allow each to evolve independently using Glauber dynamics. We measure the spin overlap between the systems as a function of time and refer to this overlap as the “heritability.” It is interesting to note that our notion of heritability is in some sense the opposite of that in “damage spreading,” where two slightly different initial states evolve according to the same dynamical realization [11–14].

Our key finding is that heritability, as embodied by the spin overlap between twins, decays as a power law in time. For finite systems, an absorbing state is always reached and we measure the average value of the overlap between the twins in their final states. We find that this quantity decays as a power law in system size. Based on a finite size scaling ansatz we obtain a relationship between the power law in system size for finite systems and the power law in time for infinite systems. This relationship is observed to hold in our numerical results.

Methods. We carried out simulations of the 2D ferromagnetic Ising model on an \( L \times L \) square lattice with periodic boundary conditions. Each site carries a spin \( S_i = \pm 1 \), \( i = 1, 2, ..., N \), where \( N = L^2 \). The energy \( E \) of the system is \( E = -\sum_{<i,j>} S_i S_j \), where the sum is over nearest neighbor pairs.

The system evolves through zero-temperature Glauber dynamics. An elementary step consists in choosing a random site \( i \) and computing the energy change \( \Delta E \) of flipping the spin \( S_i \). If \( \Delta E < 0 \) the spin is flipped (\( S_i \leftarrow -S_i \)); if \( \Delta E = 0 \) the spin is flipped with probability \( 1/2 \); and if \( \Delta E > 0 \) the spin is not flipped. These moves are repeated until the lattice reaches a final state where no spin flips are possible. This absorbing state is either one of the two homogeneous ground states or a striped state [5]. A striped state has one (or more) vertical or horizontal stripes (but not both) whose boundaries constitute domain walls separating regions of antiparallel spin orientation. The probability that the absorbing state has a single stripe is shown to be \( 0.339 \ldots \) for large \( L \), while the probability of multiple stripes is very small [5, 7].

To distinguish the influences of nature and nurture, we simulate a pair of Ising lattices with identical initial conditions (“identical twins”). We study the effects of a deep quench, in which the initial state is at infinite temperature (each spin is independently chosen by a coin toss). The subsequent application of zero-temperature Glauber dynamics then effects an instantaneous quench from infinite to zero temperature. Each twin evolves independently according to zero-temperature Glauber dynamics until it reaches an absorbing state. The time \( t \) is measured in sweeps with one sweep corresponding to \( N \) spin-flip attempts. At each time step \( t \), we study the overlap \( q_t(L) = \frac{1}{N} \sum_{i=1}^{N} S_{i1}(t)S_{i2}(t) \) between the twins, where \( S_{i1}(t) \) denotes the state of the \( i \)th spin at time \( t \) in twin 1, and similarly for \( S_{i2}(t) \). The influence of initial conditions is quantified by \( q_0(L) \), where \( q_0(L) = 1 \) for any \( L \). We are interested in the average \( q_t(L) \) over both initial conditions and the subsequent dynamics and, in particular, in understanding both the size dependence of final overlap, \( \overline{q}_\infty(L) = \lim_{t \to \infty} q_t(L) \) and the time dependence of the infinite volume limit \( \overline{q}_\infty \). As we shall see, the behavior of \( \overline{q}_\infty(L) \) and \( \overline{q}_\infty \) are connected by a finite size scaling ansatz.

We studied 21 lattice sizes from \( L = 10 \) to \( L = 500 \). For each size we study 30,000 independent pairs of twins out to a time such that almost all systems are in an absorbing state. For each initial condition we compute only two dynamical trajectories, one for each twin. This approach is statistically equivalent and more efficient than averaging over both the dynamics and initial conditions. From \( q_t(L) \) for each pair of twins, we compute the mean \( \overline{q}_t(L) \).

Results. Figure 1 shows a log-log plot of \( \overline{q}_t(L) \) vs. \( t \) for several \( L \). We observe that for short and intermediate times, \( \overline{q}_t(L) \) appears to follow a single curve for all \( L \), until an \( L \)-dependent time scale when \( \overline{q}_t(L) \) separates from the main curve and a plateau is reached. It is reasonable to suppose that the single curve represents the infinite volume behavior \( \overline{q}_\infty \) to good approximation. The initial decay of \( \overline{q}_\infty \) is rapid, followed by a shoulder that goes to about \( t = 100 \). The subsequent behavior appears to be described by a power law. A power law fit of the form \( \overline{q}_\infty \propto t^{-\theta_0} \) for the largest two sizes, \( L = 400 \) and \( L = 500 \), from \( t = 500 \) to \( t = 10^4 \) yields \( d = 0.59(3) \) and \( \theta_0 = 0.216(7) \) for \( L = 400 \), and \( d = 0.62(3) \) and \( \theta_0 = 0.225(6) \) for \( L = 500 \). The error bars are obtained by the bootstrap method. Based on these two sizes, we estimate that the heritability exponent describing the decay of the overlap with time is \( \theta_0 = 0.22 \pm 0.02 \).
Next we discuss $q_\infty(L)$, the finite size behavior of the absorbing value of $q_t(L)$. For $L < 300$ we simulate all systems until they are absorbed, but for the largest several sizes this is not possible. Therefore, for the small fraction of pairs with at least one unabsorbed twin, we approximate their contribution to $q_\infty(L)$ by the value of the $q_t(L)$ at the largest $t$, which is on the plateau for all sizes. The justification for this approximation is that the plateau value of $q_t(L)$ is nearly equal to $q_\infty(L)$ for two reasons. First, most twins are absorbed before the plateau is reached. The second reason is more subtle. On the plateau essentially all unabsorbed systems are in a diagonal stripe state. The diagonal stripe, appearing in about 4% of systems, is the longest lived metastable state with a lifetime that scales as $L^3$ [5]. A diagonal stripe decays by a random walk in the width of the stripe and almost always reaches one of the two homogeneous ground states. A simple random walk argument shows that the probability of the final state being all $+1$ is equal to the fraction of $+1$ spins in the diagonal stripe state. Thus the expected value of $q_\infty(L)$ averaged over the dynamics is approximately equal to $q_t(L)$ on the plateau, which justifies the approximation. For the largest size studied ($L = 500$), approximately 8% of pairs of twins are not yet absorbed at the end of the simulation. This fraction is to be expected if all unabsorbed pairs have a twin in a diagonal stripe state. Comparison between this approximation and the exact simulation for smaller sizes shows that the error due to the approximation is much smaller than the statistical errors due to the finite sample size.

The results for $q_\infty(L)$ are shown in Fig. 2. We find that the data are well fit by a power law of the form $q_\infty(L) = aL^{-b}$. We performed a series of power law fits in which we successively dropped smaller sizes. The only significant deviation from a power law fit occurs if the data from the smallest size, $L = 10$, are included. The quality of the fits is good and the results independent of the minimum size within the error bars indicating that the corrections to scaling are small. For example, if $L = 10$ and 20 are excluded the best fit result is $a = 0.81(3)$, $b = 0.46(1)$ with $\chi^2$/d.o.f. $= 0.51$ and the corresponding quality of fit is $Q = 0.95$. Our best estimate of $b$, taking into account both statistical and possible systematic errors, is $0.46 \pm 0.02$.

The exponents $b$ and $\theta_h$ can be related via a finite-size scaling ansatz. During coarsening, the typical domain size $\xi$ grows as $\xi \sim t^{1/z}$ where $z$ is the dynamic exponent for coarsening and $z = 2$ for zero temperature Glauber dynamics [4]. We therefore postulate the following finite size scaling form for $q_t(L) \sim t^{-\theta_b}f(t^{1/z})$, where the function $f(x)$ is expected to behave as:

$$f(x) \sim \begin{cases} 1 & \text{for } x \ll 1, \\ x^{2\theta_h} & \text{for } x \gg 1. \end{cases}$$

The large $x$ behavior is required to ensure that $q_t(L)$ approaches a constant for large $t$ and finite $L$. In particular, the $t \to \infty$ behavior is $q_\infty(L) \sim L^{-2\theta_h}$, so that $b = z\theta_h = 2\theta_h$, which agrees within errors with our numerical results $b = 0.46 \pm 0.02$ and $\theta_h = 0.22 \pm 0.02$. Since $b$ is obtained from the data for many sizes while $\theta_h$ is obtained from a limited range in $t$ and only two sizes, we consider $b$ to be a more reliable value. We also considered initial conditions with random spins but fixed magnetization rather than infinite temperature. If the fixed magnetization is within $1/\sqrt{N}$ of zero, the results for $b$ are the same as for the infinite temperature case.

Heritability and persistence. Heritability is related, at least superficially, to the phenomena of persistence. In the context of phase ordering kinetics, persistence is defined as the fraction of spins that are unchanged from their initial values at time $t$. This quantity is found to decay as a power law and the exponent $\theta_p$ is called the “persistence exponent.” Numerical simulations on the
2D Ising model with zero temperature non-conserved dynamics yield [9] $\theta_p = 0.22$ (with no error bars quoted) and [15] $\theta_p = 0.209(2)$ (error bars are only statistical). These numbers are within the error bars of the heritability exponent $\theta_h = 0.22 \pm 0.02$ obtained here.

In addition, our exponent $b = 0.46 \pm 0.02$ describing the finite size decay of heritability can be compared directly to the finite-size persistence exponent $\theta_{\text{Ising}} = 0.45 \pm 0.01$ [16]. (As discussed in [17], the same finite scaling arguments that show that $b = z\theta_h$ demonstrate that $\theta_p$ defined in [9] and $\theta_{\text{Ising}}$ defined in [16] are related by $\theta_{\text{Ising}} = z\theta_p$.)

In contrast, for the one-dimensional ferromagnetic Ising model one can compute analytically both the persistence exponent and the heritability exponent using the mapping from zero-temperature Glauber dynamics to the voter model and to coalescing random walks (see, e.g., [18, 19]). It is shown in [10, 18] that $\theta_p = 3/8$, but related arguments can be used to show that $\theta_h = 1/2$ and $b = 1$. While the persistence and heritability exponents are distinct in one dimension, it may be that they are exactly the same in two dimensions or it may be that they are simply close but not identical.

Discussion. Motivated by general questions concerning the related phenomena of local nonequilibration, chaotic time dependence and predictive power of initial configurations in stochastic dynamics, we studied the simple 2D homogeneous Ising ferromagnet with an initial state quenched from $T = \infty$ to $T = 0$. In analogy to the classic approach to nature (initial configuration) vs. nurture (dynamics), we performed Monte Carlo studies of identical twins, $S^1(t)$ and $S^2(t)$, raised in independent dynamical environments with system sizes up to $L = 500$.

The quantity we focused on was the overlap $q(L)$ between $S^1(t)$ and $S^2(t)$, which was then averaged over 30,000 sets of identical twins to give $\overline{q(L)}$. Extensive studies of the asymptotic behavior of $\overline{q(L)}$ for large $t$ and $L$ were performed. Our main conclusions are as follows:

1. For finite $L$, there are limiting absorbing states $S^j(\infty)$ and overlaps $q_\infty(L) \sim aL^{-b}$ with $b = 0.46 \pm 0.02$.

2. $\overline{q(L)}$ appears to approach an infinite volume limit $q_\infty$ as $L \to \infty$ with $q_\infty \sim dt^{-\theta_h}$ and $\theta_h = 0.22 \pm 0.02$.

3. Finite size scaling considerations suggest that $b = 2\theta_h$, consistent with our numerically estimated values and with the exact 1D values.

4. The numerical values of the heritability and persistence exponents are very close for the 2D Ising model, though these exponents are distinct in one dimension: for the 1D Ising model the heritability exponent is larger than the persistence exponent.

5. Since $\theta_h > 0$, the 2D Ising ferromagnet displays weak LNE. However, given the smallness of $\theta_h$, information about the initial state decays rather slowly, persisting for long times.

It would be interesting to study heritability both in higher dimension and in disordered systems and to understand the relationship between persistence and heritability in these situations.

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