Non-Boussinesq effect: Thermal convection with broken symmetry

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We investigate large Rayleigh number ($10^6$–$10^9$) and large Prandtl number ($10^2$–$10^3$) thermal convection in glycerol in an aspect ratio one cubic cell. The kinematic viscosity of the fluid strongly depends upon the temperature. The symmetry between the top and bottom boundary layers is thus broken, the so-called non-Boussinesq regime. In a previous paper Wu and Libchaber have proposed that in such a state the two thermal boundary layers adjust their length scales so that the mean hot and cold temperature fluctuations are equal in the center of the cell. We confirm this equality. A simplified two-dimensional model for the mean center temperature based on an equation for the thermal boundary layer is presented and compared with the experimental results. The conclusion is that the central temperature adjusts itself so that heat fluxes from boundaries are equal, temperature fluctuations at the center symmetrical, at a cost of very different temperature drops and Rayleigh number for each boundary. © 1997 American Institute of Physics.

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I. INTRODUCTION

The classical problem of Rayleigh-Bénard convection is often considered in the Boussinesq approximation: a fluid of constant material properties is assumed, except for a variable density in the buoyancy term. The parameters in the problem are the Rayleigh number $Ra$, defined as $Ra = \alpha g \Delta T L^3 / \nu k$, where $\alpha$ is the thermal expansion coefficient, $g$ the gravitational acceleration, $\Delta T$ the temperature drop, $L$ the depth of the fluid, $\nu$ the kinematic viscosity and $k$ the thermal diffusivity, and the Prandtl number $Pr = \nu / k$. Above a critical $Ra$, the buoyancy force overcomes the dissipative effects of viscous drag and heat diffusion, and convective motion sets in.\(^1\)

The early works in this field were mostly focused on the onset of convection. In recent years, significant advances have led to the introduction of different turbulent states,\(^2\)-\(^5\) far beyond the onset of convection, with various scaling laws. In those studies the Boussinesq approximation holds,\(^2\),\(^6\) and there is a top-bottom symmetry in the velocity and temperature fields.

In this paper, we describe a thermal convection experiment using glycerol as the working fluid. Glycerol’s viscosity decreases dramatically as the temperature is increased, which produces non-Boussinesq effects and a top-bottom asymmetry. In our experiment, we use glycerol with a purity better than 99.5% in weight. Its viscosity varies from 14.4 to 2195 centiPoise in the temperature range of 100–15 °C where the experiments were performed.\(^7\) The thermal conductivity is roughly constant. The Prandtl number ($Pr = \nu / k$) varies from 300 to 7000, so we have a high Prandtl number regime compared with previous experiments in gases.\(^2\)-\(^5\)

In Secs. II and III, we focus on the experimental setup, the temperature measurements, and the visualization of the flow. Because of the asymmetric temperature profile, the temperature at the central region is skewed toward the warmer bottom value. Experimentally, we confirm that a dynamic equilibrium can be reached, as was previously found by Wu and Libchaber.\(^5\)

In Sec. IV, we present a two-dimensional model and use it to determine the central mean temperature in terms of boundary temperatures. The model assumes comparable heat fluxes from top and bottom thermal sublayers impinging into a central mixing region. We thus relate the bulk temperature to the asymmetry of the vertical structure within these thin thermal layers.

II. EXPERIMENTAL SETUP

In Fig. 1, we show a diagram of the experimental apparatus. The convection cell is a cubic box of characteristic length 183 mm. The sides are made of 5 mm thick acrylic glass. The top and bottom plates are made of aluminum, thickness 25 mm and 15 mm, respectively, with careful horizontal positioning. A Kapton foil heater was used and glued with good thermal contact to the bottom plate. A Kepco dc power supply with a 0.01 mV stability was used for the heater. The heating power ranged between 20 and 250 W. The top plate was cooled using water flowing in grooves machined through the upper side of the top plate (6 liters per minute). The temperature of the cooling water was regulated by a Neslab water circulator. Throughout the experiment, the temperature stability of the top and bottom plate was better
than 0.04 °C and 0.2 °C, respectively. The control parameter Ra was changed by adjusting the top temperature and the heating power.

Plate temperature was measured with a thermal resistor (thermistor). Two thermistors were inserted inside each plate using epoxy with high thermal conductivity, and then calibrated. Temperature measurements within the fluid were done differently. Two movable thermistors were mounted on a 1 mm diameter stainless-steel tube, using a stepping motor to allow vertical translation of the detectors in the fluid. With these thermistors, we measured the top and bottom boundary layers and the temperature fluctuations in the central region of the convection cell. The physical size of a thermistor is 0.8 mm in length and 0.4 mm in diameter, and the thermal response time is about 150 ms. The two probes could be positioned with an accuracy of 0.5 mm, one pointing downwards and the other one upwards, in order to span both boundary layers. The leads were put through the stainless-steel tube. We used a Wheatstone bridge and a EG&G 5210 lock-in amplifier to detect the temperature signals. The working lock-in frequency was 2 kHz and the current through the thermistors was kept less than 0.05 mA to avoid heating effects. The balance bridge consisted of resistors connected in series ranging from 10 to 90 000 Ω. We used a HP3562 dynamical signal analyzer to sample the voltage signal from the lock-in amplifier. The time series was processed by a HP work station. The typical sampling rate varied from 5 to 10 Hz; the analog-digital converter had 16 bits resolution. We took typically 20 min of data at each vertical position in the boundary layer. To obtain a smooth histogram (PDF) in the central region of the cell, more that 6 h of sampling was needed.

To visualize the flow, we used a shadowgraph technique. From a diaphragm and a collimating lens, a nearly parallel light beam was sent through the cell to a screen located 20 cm from the cell. Velocity information on the flow structure was extracted from video recordings.

### III. MEASUREMENTS

A typical temperature profile across the cell is shown in Fig. 2, the inset being a blowup of the top boundary layer profile. Near the wall, heat is transported essentially by thermal conduction, and this region defines the thermal boundary layer. As the thermal conductivity of the fluid is almost a constant throughout the entire temperature range used in our experiment (2.7% change over 100 K), the temperature profile of the thermal boundary layer is linear close to the wall. It can be well fitted by the function

\[ T_c - T_b = \frac{1}{m_2} \tan \left( m_2 z \right) \]

where \( T_c, T_b, \) and \( T_t \) are the temperature at the center, top plate, and bottom plate, respectively. The thermal boundary layer thickness is traditionally defined as \( \lambda = 1/m_2 \). That is, \( \lambda \) is defined as the distance at which the extrapolation of the linear part of the profile equals the central mean temperature. Between the two top and bottom thermal layers lies the central region of the cell. The temperature drop across this central region is extremely small. That is, the entire convective cell, to within a small distance from the horizontal boundaries, is a region of essentially constant mean temperature. The effective heat transfer, due to the advective motion of the fluid, creates a kind of thermal “short circuit.”

In the Boussinesq case, the mean temperature of the central region will lie midway between the top and bottom wall temperatures. In our experiment, as shown in Fig. 2, this temperature lies closer to the bottom temperature, i.e., the fluid is on average warmer. This asymmetric structure is an important feature of convection in glycerol. Table I is a compendium of the main data of this experiment.
TABLE I. Experimental values of Rayleigh number Ra; central region temperature $T_c$; central region viscosity $\nu_c$; and heat diffusivity $\kappa_c$; top and bottom temperature drop $\Delta_{l,b}$; thermal layer thickness $l_{l,b}$; product of central region viscosity and plates viscosity $(\nu_l\nu_b)^{1/2}$.

<table>
<thead>
<tr>
<th>$Ra$</th>
<th>$T_c$ [°C]</th>
<th>$\nu_c$ [cm$^2$/s]</th>
<th>$\kappa_c$ [cm$^2$/s]</th>
<th>Region</th>
<th>$\Delta_l$ [K]</th>
<th>$\lambda_l$ [mm]</th>
<th>$(\nu_l\nu_b)^{1/2}$ [cm$^2$/s]</th>
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<tr>
<td>$5.31\times 10^8$</td>
<td>69.8</td>
<td>0.417</td>
<td>0.000915</td>
<td>$t$</td>
<td>39.8</td>
<td>1.608</td>
<td>0.281</td>
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<td>0.382</td>
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<td>$b$</td>
<td>22.7</td>
<td>1.261</td>
<td>0.260</td>
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<td>0.459</td>
<td>0.000917</td>
<td>$t$</td>
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<td>2.058</td>
<td>0.299</td>
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<td>0.000918</td>
<td>$b$</td>
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<td>2.638</td>
<td>0.317</td>
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<td>0.000918</td>
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<td>6.433</td>
<td>1.772</td>
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<td>16.9</td>
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<td>11.876</td>
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<td>1.044</td>
<td>0.000932</td>
<td>$t$</td>
<td>51.3</td>
<td>3.304</td>
<td>0.312</td>
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<tr>
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<td>0.456</td>
<td>0.000916</td>
<td>$b$</td>
<td>12.9</td>
<td>1.763</td>
<td>0.874</td>
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</table>

Since our measurements of the boundary layer thickness are made at only one physical point, it is interesting to check the relation between this measured thickness and the imposed temperature drop across the entire layer. Figure 3 shows, for all Ra, that the ratio between the two boundary layer thicknesses is very close to the ratio between the temperature drops across them. It confirms that the heat flux across the layers is essentially constant, we have $\Delta_l/\lambda_l=\Delta_b/\lambda_b$. Later in this paper, $\chi$ will be defined as the ratio$^3$ between the bottom and top temperature drops, $\chi=\Delta T_b/\Delta T_t$. This parameter, here equal to $\lambda_l/\lambda_t$, characterizes the asymmetry, and is thus equal to one in the Boussinesq case.

In the Boussinesq case, moreover, the two thermal layers have the same thickness. Without a large scale flow, the two layers are disconnected, $\lambda$ is independent of the size of the box $L$, which leads to the scaling law $\lambda/L=Ra^{-1/3}$. In the presence of a large scale circulation, the scaling is altered, the box size enters, and the exponent of the power law becomes $-2/7$.

Once the top to bottom symmetry is broken, the main questions we face are: what is the mechanism for adjusting the thickness of the thermal boundary layers, and what are the temperature drops across the layers?

For each thermal boundary layer, one can define an effective Rayleigh number $Ra_e$ based upon local thermal properties of the fluid and the temperature drop across the layer, where $Ra_e=(\alpha g \Delta T l^3/\kappa \nu)$ and $\Delta T_l$ is the temperature drop at the boundary. Since $\theta=\nu k\alpha g l^2 \Delta T_l$ has the dimension of a temperature, we call this parameter the temperature scale of the boundary. In the above expressions the values for $\kappa$, $\alpha$, and $g$ are essentially temperature independent. For $\nu$, we used the value $\nu=1/[(T_1+T_c)/2]$, where $T_1$ and $T_c$ are the

![FIG. 3. (KΔλ)/(KΔλ) vs Rayleigh number. $K$ is the heat conductivity.](image-url)
temperatures at the plates and in the central region, respectively.

The central region temperature fluctuations are caused by plumes pinching off from the boundaries and $\theta$ represents their mean value. It was observed in helium that, in the central region, the histogram of temperature fluctuations was symmetrical. It was then proposed\(^5\) that the two thermal layers adjust their thicknesses, so as to make the temperature scales $\sim u_5 = n_k / \alpha g l^3$ equal. This is the physical origin of the proposed scaling.

Let us now go back to the experiment with glycerol. First, as observed in helium, the histogram of temperature fluctuations measured in the central region of the cell has a symmetrical form $\sim \text{Fig. 4}$. Second, let us plot the ratio of two quantities for the thermal layers, the temperature scale $u$ and the effective Rayleigh number $Ra$ $\sim \text{Fig. 5}$. The temperature scale ratio is clearly constant for all values of $x$. This confirms the scaling law proposed, equality of the two boundary layer temperature scales. This is one of the main results of the paper. A possible scaling proposal, equality of the two boundary layer Rayleigh numbers, is not observed.

We consider now the velocity field at the boundaries. The temperature time series, irrespective of the position of measurement, always shows a high-frequency cutoff $f_c$. Tilgner \textit{et al.}\(^\text{10}\) showed that there is a direct link between the velocity profile and the profile of the cut-off frequency $f_c$ of the temperature time series. The cut-off frequency in the power spectrum is related to the velocity by $f_c = U / \Lambda$, where $\Lambda$ represents the length scale of the thermal structures; we assume $\Lambda = \lambda$. A high cut-off frequency is a result of the high velocity $U$ and the small-scale thermal structures. Thus the $f_c(Z)$ curve qualitatively represents the amplitude of the velocity field, where $Z$ represents the vertical coordinate measured from the plates. As the average horizontal velocity is about 10 times larger than the average vertical one,\(^\text{10}\) the cut-off frequency gives important information about the horizontal velocity. This method has also been used in more recent experiments in gases\(^\text{11}\) and mercury.\(^\text{3}\)

In Fig. 6, we plot the dependence of $f_c$ on the distance from both the bottom and top plate at $Ra = \wedge.1 \times 10^8$. $f_c$ increases first linearly near each plate, reaches a broad maximum and then decreases. The symmetry of the velocity field is in fact a consequence of the constant mean properties of the central convection field. The thickness of the viscous boundary layer is greater than that of the thermal boundary layer, by a factor of 2–11. Thus the asymmetry of the thermal layers does not affect the velocity layers, which in effect live in an isothermic fluid. But as $\Lambda$ is different for the two layers, so is the amplitude of the maximum value of $f_c$.

The relatively thick velocity layer is not in itself a source of the non-Boussinesq effects. However, because of the variation of viscosity with temperature, there is a kind of momentum sublayer associated with each thermal layer, which can be computed under the condition of constant viscous stress. This last assumption, which we shall use in the next section for explicit calculations, again depends on the high Prandtl number and the relative thinness of the thermal layers.

Visualization in fluids can often give valuable insights. Figures 7(a) and 7(b) show shadowgraphs taken with a conventional camera. The exposure time for Fig. 7(a) is 1/15 s. Active thermal plumes are visible on the comparatively quiet background. For Fig. 7(b), the exposure time is 15 s. The
flow can then be seen to include a large scale current. In both cases, the Rayleigh number value was \( 2.3 \times 10^8 \). There is clearly a flow rotating counter-clockwise, which drags the upwelling and downwelling thermal plumes away from the locations where they were generated. In the center of the cell, the fluid is often left relatively quiet with occasional passage of thermal plumes coming from both plates. The large-scale-flow rises from one corner of the cell and descends along the diagonally opposite corner.

This large scale, coherent flow can be furthermore confirmed by a three-dimensional visualization method, which was newly employed in our experiment. Using a conventional shadowgraph technique, the thermal structures are projected onto a two-dimensional screen. The knowledge about the depth is lost. 3D visualization is realized using two beams of color coded light. Two separated beams pass through green and red color filters. The angle between the two beams is about \( 5^\circ–8^\circ \). The two sets of shadowgraphs overlap slightly on the screen. An observer using glasses equipped with the same set of filters, but in opposite order, recovers the three-dimensional view. No quantitative results were obtained. However, the method allows geometrical and spatial representation of the large scale flow, and of thermal plumes coming off the boundaries.

For our fluid, the high Prandtl number insures that the diffusion of momentum dominates over the diffusion of heat. The thermal plumes are therefore persistent and can extend across the cell. The deformation of thermal structures occurs by collision or by stretching. The cap diameter of the plumes does not seem to grow significantly over the course of the rising or descending period.\(^{12,13} \) The speed strongly depends on the surrounding flow structures. A following plume tends to move in the predecessor’s wake. Plumes moving in the direction of the coherent flow have faster speed than those moving against it. Figure 8 presents the vertical velocities of the hot and cold plumes measured at midheight of the cell, using taped video records of the shadowgraph, as a function of Ra. The solid line in the plot shows a \( 1/2 \) slope, as observed for the large scale flow.\(^6 \)

In Fig. 9, we show the Nusselt number versus Rayleigh number, a power-law fit indicating that our experiment falls into the hard turbulent region. The experimental data are collected in Table I.

![FIG. 7. (a) Shadowgraph taken at Ra=2.3\times10^8. The hot and cold plumes and the large scale flow are visible. The plumes appeared connected to the boundary. (b) Same as (a) but over a time interval of 15 s, visualizing the large scale flow. Shadow of the stainless tube and the two detectors, hanging from the top plate, visible.](image)

![FIG. 8. Vertical velocities for hot (dark) and cold (open) plumes vs Ra. Velocity measurements were made through shadowgraphs, for plumes in the midheight region of the convection cell.](image)

![FIG. 9. Nu vs Ra. Nusselt numbers were calculated from the thermal boundary-layer thickness, \( \text{Nu} = L/(\lambda + \alpha)_p \). The solid line is the best fit.](image)


IV. TWO-DIMENSIONAL MODEL

In this section we consider a model allowing an analytical calculation of the average temperature over the central region of the convection cell, directly from a thermal sublayer. The basic physical mechanism of turbulent convection is fairly clear: owing to the pattern of warm buoyant plumes and cool descending plumes, convection currents are setup in the central region which tend to thin the thermal boundary layers at the walls and enhance transport heat away from the walls and eventually into plumes. Non-Boussinesq effects arise when the thinning process is affected by temperature dependence of the fluid viscosity.

At large Rayleigh number this process can be studied analytically in two dimensions, under steady conditions, by concentrating on a single convection cell and calculating in detail the boundary-layer structure. In the present problem, a more reasonable assumption is that (A) intermittent plumes transport from wall to central region, where mixing occurs, but not between top and bottom thermal layers. This is approximately true in our experiment, since one consequence of an independence of layer thickness is an $Ra^{1/3}$ power law, and this is approximately observed in the absence of large-scale flow. Laminar cells yield an $Ra^{1/5}$ law in our case.

A second, related assumption is that (B) the pattern of convection currents within the central region is essentially symmetric top and bottom. Then, a fundamental constraint is the equality between the average heat flux at each wall. That is, layers formed on each wall from identical convection currents should carry away equal and opposite heat. If the fluid properties are independent of temperature, this is achieved when the central temperature is the average of the wall temperatures. If they are not, the central temperature is assumed to adjust to make the fluxes agree.

To formulate this model in the simplest setting we consider two dimensions and replace an intermittent time-dependent problem by a steady one. Let $H$ be the horizontal extent of a top or bottom boundary layer, terminating in a rising or descending plume (Fig. 10). Since $H$ represents the extent of horizontal motion of fluid warmed at the bottom or cooled at the top boundary, it is roughly a horizontal cell size in the usual sense of Rayleigh-Bénard convection. $H$ can also be viewed as a typical horizontal distance between neighboring descending and ascending plumes. Thus, in our experiments $H$ is much smaller than the cell height $L$. In our high Prandtl number experiments, the layer consists of a viscous layer of thickness $\lambda_T$, and a thermal sublayer of thickness $\lambda<\lambda_T$. To check this, note first that, if $U$ is a typical velocity at the outer edge of the viscous layer, a rate of strain $\sim U/\lambda_T$, is developed in the viscous layer, the resulting stress $\nu U/\lambda_T$, will be maintained in any sublayer. So $U_\lambda \sim U\nu/\lambda_T$, is a typical velocity in the thermal layer. From the classical Prandtl theory we also have $\lambda < \sqrt{\nu/\rho H/\lambda}$, where $\lambda$ is a typical horizontal extent of a cell of rising and falling fluid. Hence, a basic scaling is $k\lambda_T$, which is an estimate of the typical cell size in our experiment. Thus, from Table I we see that these estimates indicate values of $\lambda_T/\lambda$ in the range 2–11, the lower value corresponding to the smaller horizontal extent of a top or bottom boundary layer. Thus

\[
\lambda_T \approx \frac{\nu}{\rho H/\lambda}
\]

where $\lambda$ is a wall strain rate and $\nu = \nu(T_c)$. By assumption $B$, the same $\lambda$ will be used for both upwelling and downwelling plumes. Using the stream function appropriate to an incompressible flow, $(u, v) = (\psi, -\kappa \psi)$, we may then solve for $u$ as a function of von Mises variables $x$, $\psi$ in the form

\[
u(T) \frac{\partial u}{\partial y} = \nu \frac{\partial g(x)}{\partial y}
\]

where $g(x)$ is a wall strain rate and $\nu = \nu(T_c)$. By assumption $B$, the same $g(x)$ will be used for both upwelling and downwelling plumes. Using the stream function appropriate to an incompressible flow, $(u, v) = (\psi, -\kappa \psi)$, we may then solve for $u$ as a function of von Mises variables $x$, $\psi$ in the form

\[
u(T) \frac{\partial u}{\partial y} = \nu \frac{\partial g(x)}{\partial y}
\]

The thermal boundary-layer equation is

\[
u \frac{\partial u}{\partial y} + v \frac{\partial T}{\partial y} - \kappa \frac{\partial^2 T}{\partial y^2} = 0.
\]

To pass to von Mises variables we use
temperature is then

\[ \frac{\partial}{\partial y} = \frac{u}{\partial \psi_x}, \quad \frac{\partial}{\partial x} = \frac{\partial}{\partial \phi} \frac{\partial}{\partial y} - \frac{v}{\partial \psi_x}. \]  

(3)

In von Mises variables the boundary-layer equation for the temperature is then

\[ \frac{\partial T}{\partial x} - \kappa \frac{\partial}{\partial \phi} \left( q \frac{\partial T}{\partial \psi} \right) = 0. \]  

(4)

With \( d\sigma = \sqrt{2g(x)} dx \), \( \sigma(0) = 0, \sigma(L) = \sigma_L \), we now solve the boundary layer equation (1) with conditions

\[ T(0,\psi) = T_c, T(\sigma,0) = T_{i,b}, T(\sigma,\infty) = T_t \]  

(5)

on \( 0 < \sigma < \sigma_L \), and evaluate the flux \( Q = \int_0^\infty (T(A_{1,b},A) - T_c) ds \). We then adjust \( T_c \) until the fluxes at the top and bottom layers are equal and opposite, thus invoking our assumption \( A \).

Since Eq. (1) is nonlinear we began by using a numerical scheme based on a nonlinear Crank-Nicholson algorithm. The temperature dependence of the fluid properties of glycerol was accounted for only in the kinematic viscosity, using standard formulas.7 In this way we could study both the development of a boundary layer from fresh fluid at the central temperature (the calculation implicit in our assumptions) as well as circulating boundary layers, where after traversing a fixed distance the layer profile becomes the initial profile for the opposite wall layer, and so on until a steady equilibrium results.

For accurate calculations of the boundary layer development from fluid at a uniform central temperature a better procedure is to solve Eq. (2) directly from the similarity form

\[ T - T_c = F(\eta), \eta = \psi/\sigma^{2/3}, \quad d\sigma = \kappa \sqrt{2g(x)} dx. \]  

(6)

This yields

\[ \frac{2 \eta F'}{3} + \left( \sqrt{\int_0^{\eta} \frac{v_c}{\eta(T_c + F(s))} ds} F' \right)' = 0. \]  

(7)

It is useful to rewrite this as an equation for \( G(\eta) = \sqrt{\eta F'} \), in the form

\[ (2/3) \sqrt{\eta G} + G' + [\epsilon(G) G]' = 0, \]  

(8)

where

\[ \epsilon(G) = \frac{1}{\sqrt{\eta}} \int_0^{\eta} \frac{v_c}{\eta(T_c + F(s))} ds - 1. \]  

(9)

Multiplying Eq. (3) by \( \phi(\eta) = \exp(4\eta^{3/2}/9) \) and integrating from 0 to \( \eta \) we obtain

\[ G(\eta) = -\frac{2}{3} Q \phi^{-1} + \frac{2}{3} Q \epsilon(G) \phi^{-1} \]  

\[ + \frac{\phi^{-1}}{1 + \epsilon(G)} \int_0^{\eta} G(s) \phi(s)ds. \]  

(10)

Here \( Q = -(3/2) \sqrt{v_c/v_{i,b}} G(0) \) is the heat flux in the boundary layer at \( \sigma = 1 \).

To obtain the central temperature as a function of wall temperatures, we first choose a \( T_c \), then solve Eq. (10) iteratively (starting by retaining only the first term on the right), first with \( Q = q \) and then with \( Q = -q \), for some given \( q \). Integrating \( G \) in each case gives, \( T_{b-c} = T_{i-c} \) and \( T_{t-c} \), respectively, and then the two wall temperatures.

We now describe results and compare the calculation with the experimental data. We first note that in general \( T_c \) is a function of both \( T_i \) and \( T_b \) separately, and various data displays are needed to suggest the functional form. We show in Fig. 11 the theoretical values of \( T_c - (T_b + T_i)/2 \) as a function of \( T_b \) for various \( T_i \). The contours are based upon Eq. (4), but we include for comparison a curve computed from

FIG. 11. Central temperature minus average temperature as a function of bottom temperature for various top temperatures; experimental points and theory (continuous line).
the PDE directly. The two methods agree closely. The experimental data shown in Fig. 11 are for a range of upper wall temperatures and are generally slightly below the theoretical values, particularly at the higher values of \( T_b \), although the agreement is reasonably good over the entire temperature range. In Fig. 12 we show a plot against the temperature difference, with theoretical points obtained from Eq. (4) for various \( q \). The latter clearly group on lines according to \( q \), and can to some extent be collapsed by plotting against \( (T_b - T_t)/q \), but this is helpful only when \( q \) is observable.

As already mentioned, to check the effect of one boundary layer on another, we also computed, using the Crank-Nicholson algorithm, some values of \( T_c \) assuming the terminal profile of one boundary layer is the initial profile of the other, and continuing until the solution is equilibrated. It should be mentioned that for such closed convection cells the vertical layers tend to produce somewhat higher values of \( T_c \) than \( T_b \), whereas in the absence of the no-slip condition at the wall, the wall fluxes were small in the latter because of the usual boundary-layer effect. Thus the agreement is reasonably good over the entire temperature range.

In the present model the last result is a direct consequence of the boundary layer equation together with our assumptions \( A,B \) above. Indeed, from the thermal boundary-layer equation we have the estimates noted earlier, \( U_{\lambda} \sim U \alpha \nu / \lambda \nu \) and \( U_{\lambda} \sim H \kappa / \lambda^2 \). We now estimate \( U \) by equating the rate of viscous dissipation in the \( H \times L \) rectangle, \( \nu U^2 / H^2 \times LH \), to the rate of working of the plumes \( \alpha gQH \). We also use \( \lambda \nu \sim \sqrt{\nu H/U} \). Eliminating \( U_{\lambda} \) and \( U \) from these expressions, we obtain

\[
\frac{\kappa \nu}{g \alpha \lambda^3} = \left( \frac{Q^3}{g \alpha H^4 \nu} \right)^{1/4}.
\]

Since the right-hand side of this expression is the same for top and bottom boundary layers, the Wu-Libchaber assumption [Eq. (11)] follows.

Finally, we compare the data with the model of Wu and Libchaber, \( \delta \) obtained from Eq. (11) together with the approximate expression \( Q = \kappa |T_c - T_{b,1}| / \lambda \delta \), for \( Q \). In applying Eq. (11) we follow Wu and Libchaber in writing \( v_{b,1} = \nu ((T_{b,1} + T_t)/2 \), in order to interpolate between wall and center temperatures. In the temperature range of our experiments for glycerol the viscosity is roughly proportional to \( e^{-ct} \), \( c = 0.087 \). We can then solve for \( \Delta T = T_c - (T_t + T_b)/2 \) as a function of \( \Delta T = T_c - T_t \). This yields

\[
\Delta T = \frac{\Delta T}{2} \tanh(c \Delta T/12).
\]

This expression, which agrees quite well with the data, is shown in Fig. 12.
V. CONCLUDING SUMMARY

The measurements reported here, and their analysis, confirm the overall physical picture of non-Boussinesq convection described earlier. Our results are, however, for a large Prandtl number as well as large Rayleigh number. The flow pattern is characterized by long, thin plumes which span the convection field. A simple model of the heat transfer into the plumes, across the thin thermal boundary layers on the top and bottom walls, allows the calculation of the central temperature under the assumption of complete mixing. The values obtained agree reasonably well with the measured central temperature. The central temperature adjusts itself so that heat fluxes from boundaries are equal, and temperature fluctuations at the center symmetrical, at a cost of very different temperature drops and Rayleigh number for each boundary (Table I).

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8 Thermometrics thermistors, type AB6E3-GC16KA143L/37C.


