Optical solenoid beams

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Abstract: We introduce optical solenoid beams, diffractionless solutions of the Helmholtz equation whose diffraction-limited in-plane intensity peak spirals around the optical axis, and whose wavefronts carry an independent helical pitch. Unlike other collimated beams of light, appropriately designed solenoid beams have the noteworthy property of being able to exert forces on illuminated objects that are directed opposite to the direction of the light’s propagation. We demonstrate this through video microscopy observations of a colloidal sphere moving upstream along a holographically projected optical solenoid beam.

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References and links
Radiation pressure due to the momentum flux in a beam of light drives illuminated objects along the direction of the light’s wave vector. Additional forces arising from intensity gradients tend to draw small objects toward extrema of the intensity. These forces are exploited in single-beam optical traps known as optical tweezers [1], which capture microscopic objects at the focus of a strongly converging beam of light. Stable three-dimensional trapping results when axial intensity gradients are steep enough that the intensity-gradient force overcomes radiation pressure downstream of the focus. The beam of light in a tightly focused optical tweezer therefore has the remarkable property of drawing particles upstream against radiation pressure, at least near its focal point [1]. Collimated beams of light generally have no axial intensity gradients, and therefore ought not to be able to exert such retrograde forces.

In this Letter, we introduce optical solenoid beams whose principal intensity maximum spirals around the optical axis and whose wavefronts are characterized by an independent helical pitch. Figure 1 (Media 1) shows theoretical and experimentally realized examples. These beams are solutions of the Helmholtz equation, and thus propagate without diffraction [2, 3], their radial intensity profiles remaining invariant in the spiraling frame of reference [4].

Intensity gradients in a solenoid beam tend to draw small objects such as colloidal particles toward the one-dimensional spiral of maximum intensity. Radiation pressure directed by the beam’s phase gradients [5] then can drive the particle around the spiral. Under appropriate circumstances, the combination of intensity-gradient localization and phase-gradient driving can create a component of the total optical force directed opposite to the light’s direction of propagation, which can pull matter upstream along the beam’s entire length.

The vector potential for a beam of light at frequency \( \omega \) propagating along the \( \hat{z} \) direction may be written as

\[
A(r, z, t) = u(r, z) e^{-i\omega t} \hat{e},
\]

where \( k = \omega / c \) is the wave number of the light, \( \hat{e} \) is its polarization vector and \( r \) measures the two-dimensional displacement from the beam’s axis. We derive the three-dimensional optical solenoid field \( u(r, z) \) by considering the two-dimensional field \( u_0(r) \) in the plane, \( z = 0 \). Because the light propagating to \( z > 0 \) must first pass through the plane \( z = 0 \), the field in this plane completely specifies the beam. Moreover, a featureless beam imprinted with the complex field \( u_0(r) \) in the plane \( z = 0 \) will propagate into the far field as \( u(r, z) \). In this sense, \( u_0(r) \) may be considered the hologram encoding the desired beam.

Quite generally, \( u_0(r) \) may be obtained from \( u(r, z) \) by formally back-propagating the three-dimensional field to \( z = 0 \). This can be accomplished in scalar diffraction theory with the
Rayleigh-Sommerfeld formula [6],

$$u_0(r) = \int (u \otimes h_{-z})(r) \, dz,$$

where

$$h_z(r) = -\frac{1}{2\pi} \frac{\partial}{\partial z} \left( e^{ik\sqrt{r^2+z^2}} \right)$$

is the Rayleigh-Sommerfeld propagator, and where the convolution is given by

$$(u \otimes h_{-z})(r) = \int u(r',z) h_{-z}(r-r') \, d^2r'.$$

This formalism can be useful even if the desired field, $u(r,z)$, is not a solution of the Helmholtz equation, and so does not describe a physically realizable beam of light. In that case, the physical beam, $u_p(r,z)$, associated with $u(r,z)$ can be obtained by propagating $u_0(r)$ forward, again using the Rayleigh-Sommerfeld propagator,

$$u_p(r,z) = (u_0 \otimes h_z)(r).$$

Those solutions for which $|u_p(r,z)|^2$, is independent of $z$ are said to be non-diffracting [2, 3].

We now apply this formalism to designing beams of light whose intensity maxima trace out specified one-dimensional curves in three dimensions, with arbitrary amplitude and phase profiles along these curves. Such beams may be represented as

$$u(r,z) = \begin{cases} a(z) \delta(r-r_0(z)) e^{i\varphi(z)}, & z_1 \leq z \leq z_2, \\ 0, & \text{otherwise}. \end{cases}$$

Here, $r_0(z)$ is the position of the beam’s maximum at axial position $z$, $a(z)$ is its amplitude, and $\varphi(z)$ is its phase. This representation does not describe a physically realizable beam of light because it neither incorporates self-diffraction nor locally conserves energy or momentum. Equations (2) through (5) nevertheless yield a physically realizable beam that has the desired properties along $r_0(z)$, provided that self-diffraction may be neglected.

Equation (4) is most easily computed with the Fourier convolution theorem. In that case, the two-dimensional Fourier transform of $u_0(r)$ is

$$\tilde{u}_0(q) = \int_{z_1}^{z_2} a(z) e^{i\varphi(z)} e^{-iq \cdot r_0(z)} e^{-iz(k^2-q^2)^{1/2}} \, dz.$$
An inverse Fourier transform then provides \( u_0(\mathbf{r}) \), and Eq. (5) yields the associated beam of light. This result extends to three dimensions our previous descriptions of holographic line traps and holographic ring traps in the plane [7, 8].

As a step toward deriving the solenoid beam, we first consider the case of an infinite line of light propagating along the optical axis, \( \mathbf{r}_0(z) = 0 \), with uniform amplitude, \( a(z) = 1 \), but with a specified axial phase gradient, \( \phi(z) = \beta z \). For \( 0 \leq \beta \leq k \), Eq. (7) has solutions

\[
u_0(\mathbf{r}) = \beta J_0 \left( (k^2 - \beta^2)^{1/2} r \right)
\]

and \( u_\ell(\mathbf{r}, z) = u_0(\mathbf{r}) \exp(i\beta z) \), which is the zeroth-order Bessel beam [9, 10, 11]. Whereas we specified an infinitesimally finely resolved thread of light, formal back-propagation with Eq. (7) implicitly accounts for the beam’s self-diffraction. The limit \( \beta = k \) corresponds to a plane wave propagating along \( \hat{z} \). Smaller values of \( \beta \) yield more finely resolved beams that carry less momentum along \( \hat{z} \).

To create a solenoidal beam, we set \( a(z) = 1 \) and \( \mathbf{r}_0(z) = R \cos(\theta_0(z)) \hat{x} + R \sin(\theta_0(z)) \hat{y} \), where \( \theta_0(z) = z/\gamma \) is the azimuthal angle around the optical axis in a spiral of radius \( R \) and pitch \( \gamma \). In addition to establishing a spiral structure for the beam’s principal intensity maximum, we also impose a helical phase profile in the plane, \( \phi(z) = \ell \theta_0(z) \), where the helical pitch, \( \ell \), is independent of \( \gamma \). This helical phase profile will enable us to exert tunable phase-gradient forces [5] along the solenoid.

As for the Bessel beam, we seek a non-diffracting solution of Eq. (7), and so integrate over all \( z \) to obtain

\[
u_\ell(\mathbf{r}, z) = \sum_{m=\lfloor -\gamma k \rfloor}^{\lfloor \ell/k \rfloor} \frac{\ell - m}{\gamma^2} J_m(q_m R) e^{\gamma z} e^{im0} J_m(q_m r), \tag{9}
\]

where \( q_m^2 = k^2 - (\ell - m)^2/\gamma^2 \) and \( \lfloor x \rfloor \) is the integer part of \( x \). The solenoid beam thus is a particular superposition of \( m \)-th order Bessel beams. Superposition of non-diffracting modes previously has been used to synthesize multi-lobed spiral [12, 13, 14] and localized [15, 16] modes. More generally, Eq. (9) is a particular example of a rotating scale-invariant electromagnetic field [4].

Figure 1(a) shows the three-dimensional intensity distribution \( I_{\ell, \ell}(\mathbf{r}, z) = |u_{\ell, \ell}(\mathbf{r}, z)|^2 \) computed according to Eq. (9) for \( kR = 10 \), \( \theta = 30^\circ \) and \( \ell = 10 \). As intended, the locus of maximum intensity spirals around the optical axis.

Quite clearly, the intensity distribution of a solenoid depends on \( z \), and so is not strictly invariant under propagation. Nonetheless, the in-plane intensity distribution remains invariant, merely rotating about the optical axis. Such a generalization of the notion of non-diffracting propagation previously was introduced in the context of spiral waves [12]. Solenoid beams therefore may be considered to be non-diffracting in this more general sense.

Distinct solenoid beams satisfy the orthogonality condition

\[
\int u_{\ell', \ell'}(\mathbf{r}, z) u_{\ell, \ell}(\mathbf{r}, z) d^2 r dz = \delta_{\ell, \ell'} \delta(\gamma - \gamma'), \tag{10}
\]

except if \( m \equiv (\ell' - \ell')/(\gamma - \gamma') \) is an integer that falls in the range \( \max(\ell - \gamma k, \ell' - \gamma' k) \leq m \leq \min(\ell, \ell') \). This additional condition defines classes of \( m \)-congruent solenoid beams whose members are not mutually orthogonal and results from the solenoid modes’ non-trivial periodicity along the optical axis.

Figure 2 shows the effect of changing the helicity of a solenoid beam with a fixed spiral pitch, \( \alpha = \tan^{-1}(\gamma k) \). When \( \ell > 0 \), as in Fig. 2(a), the wave vector is directed along the solenoid. A particle confined to the spiral by intensity-gradient forces therefore is driven downstream
Fig. 2. Retrograde forces in a helical solenoid beam. The local wave vector $\mathbf{k}$ is normal to the light’s wavefronts, with a component in the $\hat{z}$ direction. (a) $\ell = +40$: $\mathbf{k}$ is directed along the solenoid, resulting in a downstream phase-gradient force. (b) $\ell = 0$: Wavefronts are parallel to the solenoid so that $\mathbf{k}$ is everywhere normal to the spiral. Particles trapped by intensity-gradient forces experience no net force. (c) $\ell = -40$: A component of $\mathbf{k}$ is directed back down the spiral. A particle confined to the spiral therefore moves upstream.

by this component of the radiation pressure. Changing $\ell$ does not alter $\alpha$, but changes the wavefronts’ pitch relative to $\hat{z}$. At $\ell = 0$, the wavefronts are parallel to the solenoid’s pitch. as shown in Fig. 2(b). In this case, radiation pressure is directed normal to the spiral, and so can be balanced by intensity-gradient forces. Setting $\ell < 0$ tilts the wavefronts in the retrograde direction, as shown in Fig. 2(c). The resulting reverse-sense phase-gradient force can move the particle upstream along the spiral in the negative $\hat{z}$ direction.

We experimentally projected solenoid beams using methods developed for holographic optical trapping [17, 18, 19]. In this system, a phase-only liquid crystal spatial light modulator (SLM) (Hamamatsu X7690-16 PPM) is used to imprint the hologram $u_0(\mathbf{r})$ associated with $u_{\gamma,\ell}(\mathbf{r})$ onto the wavefronts of a linearly polarized laser beam with a vacuum wavelength $\lambda = 532$ nm (Coherent Verdi). This hologram then is projected into the far field with a microscope objective lens (Nikon Plan Apo, 100×, oil immersion) mounted in a conventional inverted optical microscope (Nikon TE 2000U). The computed complex hologram is encoded on the phase-only SLM using the shape-phase holography algorithm [7]. The resulting beam includes the intended solenoid mode superposed with higher diffraction orders [20].

To visualize the projected beam, we mount a front-surface mirror on the microscope’s stage. The reflected light is collected by the objective lens, and relayed to a CCD camera (NEC TI-324AI). Images acquired at a sequence of focal depths then are combined to create a volumetric rendering of the three-dimensional intensity field [21]. The example in Fig. 1(b) (Media 1) shows the serpentine structure of a holographically projected solenoid beam with $R = 5 \mu m$.

To demonstrate the solenoid beam’s ability to exert retrograde forces on microscopic objects, we projected it into a sample of colloidal silica spheres 1.5 $\mu m$ in diameter dispersed in water. The sample was contained in the 50 $\mu m$ thick gap between a glass microscope slide and a glass no. 1 cover slip, and was mounted on the microscope’s stage. Bright-field images of individual spheres interacting with the solenoid beam were obtained with the same objective lens used to project the hologram, and were recorded by the video camera at $1/30$ s intervals. The sphere’s appearance changes as it moves in $z$ in a manner that can be calibrated [22] to measure the particle’s axial position. Combining this with simultaneous measurements of the particle’s in-plane position [22] yields the three-dimensional trajectory data that are plotted in Fig. 3 (Media 2). The gray-scale image in Fig. 3 was created by superimposing six snapshots of a single sphere that was trapped on a solenoid beam and moving along its length. This and the video sequence in Media 2 illustrate how the sphere’s image changes as it moves in $z$. 
Fig. 3. Three-dimensional trajectory of a colloidal sphere moving along one turn of an optical solenoid beam together with a multiply-exposed image of the sphere at six points in its motion (Media 2). Alternating between $\ell = \pm 30$ switches the direction of the particle’s motion relative to the propagation direction, $\hat{z}$. Arrows indicate the direction of motion for the downstream (blue) and retrograde upstream (red) trajectories.

The data plotted in Fig. 3 (Media 2) were obtained by alternately setting $\ell = +30$ and $\ell = -30$ without changing any other properties of the solenoid beam. The three blue traces show trajectories obtained with $\ell = +30$ in which the particle moved downstream along the curve of the solenoid, advancing in the direction of the light’s propagation. These alternate with two red traces obtained with $\ell = -30$ in which the particle moves back upstream, opposite to the direction of the light’s propagation. These latter traces confirm that the combination of phase- and intensity-gradient forces in helical solenoid beams can exert retrograde forces on illuminated objects and transport them upstream over large distances.

Although the solenoid beam was designed to be uniformly bright, the particle does not move along it smoothly in practice. Interference between the holographically projected solenoid beam and higher diffraction orders creates unintended intensity variations along the solenoid that tend to localize the particle. Achieving retrograde motions over distances larger than the 8 $\mu$m in our demonstration will require improved methods for projecting solenoid modes.

The foregoing results introduce solenoidal beams of light whose non-diffracting transverse intensity profiles spiral periodically around the optical axis and whose wavefronts can be independently inclined through specified azimuthal phase profiles. We have demonstrated that solenoid beams can trap microscopic objects in three dimensions and that phase-gradient forces can be used to transport trapped objects not only down the optical axis but also up. The ability to balance radiation pressure with phase-gradient forces in solenoidal beams opens a previously unexplored avenue for single-beam control of microscopic objects. In principle, solenoid beams can transport objects over large distances, much as do Bessel beams [11, 23] and related non-diffracting modes [24], without the need for high-numerical-aperture optics. Solenoid beams, moreover, offer the additional benefit of bidirectional transport along the optical axis.

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