Theory of holographic optical trapping

Bo Sun, Yohai Roichman, and David G. Grier

Department of Physics and Center for Soft Matter Research, New York University, New York, NY 10003

(Dated: December 23, 2011)

Optical traps use the forces exerted by structured beams of light to confine and manipulate microscopic objects in three dimensions. A popular implementation involves structuring the trap-forming beam with computer-generated holograms before focusing it into traps with a high-numerical-aperture optical train. Here, we present a fully vectorial theory for the forces and torques exerted by such systems.

INTRODUCTION

Holographic optical trapping (HOT) is an increasingly popular method for applying precisely controlled forces to microscopic objects [1, 2]. In this technique, a computer-generated hologram is imprinted onto the wavefronts of a laser beam using a diffractive optical element (DOE) such as a spatial light modulator (SLM). The modified beam then is relayed to a high-numerical-aperture lens, which focuses the light into the desired pattern of optical traps. This three-dimensionally structured light field influences the motions of illuminated objects through a combination of induced-dipole forces [3], which arise from local intensity gradients, and radiation pressure [4–6], which is directed by local phase gradients [7]. The former generally is referred to as the optical gradient force, and the latter as the scattering force. If the combination of these optical forces gives rise either to a mechanical equilibrium or to a dynamical steady state, the illuminated object is said to be trapped, and the structured light field constitutes an optical trap.

The forces and torques exerted by an optical trap can be computed by first calculating the electromagnetic field surrounding an illuminated object and then integrating the Maxwell stress tensor over its surface Σ,

\[ F_\mu = \oint_\Sigma T^\nu_\mu \, dS^\nu. \] (3)

where the momentum flux in the electromagnetic field

\[ F^\nu_\mu = \partial_\mu A^\nu - \partial_\nu A_\mu, \] (2)

emerges from gradients in the light’s vector potential, A. The force on a particle is then obtained by integration over its surface Σ.

OPTICAL FORCES AND TORQUES

The optical force and torque experienced by an illuminated object arises from the energy-momentum tensor \( T^\nu_\mu (\mathbf{r}) \) of the electromagnetic field. Among various formulations, the symmetric Maxwell tensor is found to simplify computation of forces and torques [15]. In SI units,

\[ \rho_0 T^\nu_\mu = -F_{\mu \rho} F^\rho_\nu + \frac{1}{4} F_{\rho \sigma} F^{\rho \sigma} \delta^\nu_\mu, \] (1)

This article presents an efficient method for computing the optical forces and torques exerted by holographic trapping patterns in practical HOT systems. Based on Debye-Wolf theory for light propagation through optical trains and the Lorenz-Mie theory of light scattering by small particles, this approach expresses the incident and scattered fields as solutions of the vector Helmholtz equa-
Similarly, the optical torque is obtained as
\[
M_\mu = \frac{1}{2} \oint_{\Sigma} \epsilon_{ijk} M^{ijk} \, dS_k
\]  
(4)

where the angular momentum tensor corresponding to the symmetric stress tensor is
\[
M_{ijk} = r_i T_{jk} - r_j T_{ik}
\]  
(5)

and where \( \epsilon_{ijk} \) is the Levi-Civita antisymmetric tensor.

In discussing the forces exerted by optical traps, it is conventional to define the trapping efficiency in the direction \( \nu \) as [8]
\[
Q_\nu = \frac{c}{n_m P} F_\nu
\]  
(6)

where \( P \) is the laser power, \( n_m \) is the refractive index of the medium, which we assume for simplicity to be homogeneous and isotropic, and \( c \) is the speed of light in vacuum. Similarly the torque efficiency for a sphere of radius \( a \) rotating about the \( \nu \) direction is
\[
\tau_\nu = \frac{\omega}{P} M_\nu.
\]  
(7)

This general formulation is common to all theoretical studies of optical trapping. Diverse implementations differ in how they treat the incident and scattered electromagnetic fields surrounding the particle, and in their strategy for performing the surface integrals in Eq. (3) and (4). The formalism described in the following sections computes the superposition of contributions from each pixel in a pixellated DOE to the overall force and torque exerted on an object in a holographic optical trapping system. Although our discussion focuses on optical trapping of isotropic homogeneous spheres, it can be generalized to particles of other shapes and compositions.

**HOT LIGHT FIELDS**

Previous discussions of the light fields projected by holographic optical trapping systems have considered only the role of the objective lens, and have treated this with scalar diffraction theory [16–20]. Although this approximation is computationally efficient and has proved effective for projecting sophisticated optical trapping patterns, its neglect of polarization effects leads to errors in predicting the projected light field. These treatments, furthermore, did not predict the forces and torques that the projected light would exert on illuminated objects. They were not suitable, therefore, for predicting and optimizing the performance of holographic trapping systems.

In a standard HOT system, the DOE is positioned in the input plane of a telescope whose role is to project the hologram onto the input pupil of the objective lens. In so doing, however, it also rotates the light’s polarization in a position-dependent manner. The image of the DOE in the objective’s input pupil thus differs from the ideal DOE in a way that affects trapping performance. Such spatially varying polarization was not considered in previous studies of the forces exerted on spheres by strongly focused light fields [9, 10]. We account for this effect by making use of recent results obtained with the Debye-Wolf formalism [21].

Because the telescope forms an image in the far field, each pixel in the DOE may be treated as a magnetic dipole. The contribution to the vector potential from the \( j \)-th pixel thus has the form [21]
\[
A_{j,\mu}^D(\mathbf{r}) = -\frac{a_j}{2\pi c} \exp (i \varphi_j) \frac{\exp (ik |\mathbf{r} - \mathbf{r}_j|)}{|\mathbf{r} - \mathbf{r}_j|} p_\mu,
\]  
(8)

where \( k = 2\pi/\lambda \) is the wavenumber of light of wavelength \( \lambda \), and where \( a_j \) and \( \varphi_j \) are the phase and amplitude at the \( j \)-th pixel, respectively. The polarization of the ray propagating along
\[
\hat{s}_1 = (\sin \theta_1 \cos \phi_1, \sin \theta_1 \sin \phi_1, \cos \theta_1).
\]  
(9)

is given by
\[
\mathbf{p} = \hat{s}_1 \times (\hat{z} \times \hat{\epsilon}),
\]  
(10)

where \( \hat{\epsilon} \) is the polarization of the incident light. The total vector potential at point \( \mathbf{r} \) is then
\[
A_{\mu}^D(\mathbf{r}) = \sum_{j=1}^{N} A_{j,\mu}^D(\mathbf{r})
\]  
(11)

for a DOE consisting of \( N \) pixels.
The image in the objective’s input pupil is a superposition of contributions from each of the DOE’s pixels. Referring to Fig. 1, a ray propagating from \( r_1 \) on the DOE in the direction \( \hat{s}_1 \) arrives at \( r_2 \) in the image plane in the direction \( \hat{s}_2 \). The angles of departure and arrival are related by the Abbe sine condition, \( f_1 \sin \theta_1 = f_2 \sin \theta_2 \), and by continuity, \( \phi_1 = \phi_2 + \pi \). All such contributions can be expressed as a superposition of plane waves through the Debye-Wolf integral [21],

\[
A^O_\mu(r_2) = \int_{\Omega_2} B_\mu(\hat{s}_2) \exp(ik\hat{s}_2 \cdot r_2) \, d\Omega_2, \tag{12}
\]

where the complex amplitude of the plane wave propagating in direction \( \hat{s}_2 \) is

\[
B_\mu(\hat{s}_2) = \frac{f_2}{2\pi c} C^\mu_\nu(\hat{s}_2) p_\nu \sum_{j=1}^N a_j \exp(i\varphi_j) \exp(ik\hat{s}_1 \cdot r_j). \tag{13}
\]

The geometric operator \( G(\hat{s}) \) accounts for rotation of the light’s polarization as it propagates through the telescope and may be represented as [21]

\[
G(\hat{s}) = \frac{\cos \theta_2}{\cos \theta_1} \mathbb{R}^{-1}(\phi_1) \mathbb{L}(\pi - \theta_2) \mathbb{L}(\pi - \theta_1) \mathbb{R}(\phi_1), \tag{14}
\]

in terms of the generalized Jones matrices

\[
\mathbb{R}(\phi) = \begin{pmatrix}
\cos \phi & \sin \phi & 0 \\
-\sin \phi & \cos \phi & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

and

\[
\mathbb{L}(\theta) = \begin{pmatrix}
\cos \theta & 0 & \sin \theta \\
0 & 1 & 0 \\
-\sin \theta & 0 & \cos \theta
\end{pmatrix}. \tag{15}
\]

**Focusing by an aberration-free objective**

The light field formed by the telescope is focused into the sample by the objective lens. The plane wave propagating in the direction \( \hat{s}_2 \) thus is transformed into a superposition of plane waves that are incident on the sample, which we assume to be immersed in a homogeneous isotropic medium of refractive index \( n_m \). The component propagating along \( \hat{s} \) with polarization \( \mu \) may be written as

\[
A^I_{\mu}(r, \hat{s}) = A^I_{\mu}(\hat{s}) \exp(i n_m k \hat{s} \cdot r) \tag{17}
\]

with a complex amplitude that also is described by a Debye-Wolf integral,

\[
A^I_{\mu}(\hat{s}) = \int_{\Omega_2} H^\nu_\mu(\hat{s}, \hat{s}_2) B_\nu(\hat{s}_2) \exp(-i n_m k \hat{s} \cdot \Delta r) \, d\Omega_2, \tag{18}
\]

where \( \Delta r = f \sin \theta_2 (\cos \phi_2, \sin \phi_2, 0) \). Assuming further that the objective lens has well corrected aberrations, the geometric tensor describing the rotation of the beam’s polarization is [22]

\[
\mathbb{H}(\hat{s}, \hat{s}_2) = \frac{i}{\pi} \sqrt{\frac{\hat{s} \cdot \hat{s}_2}{n_m}} \left(1 - \frac{1}{\hat{s} \cdot \hat{s}_2} \right)^2 \times \\
\left\{ (\hat{s} \times \hat{s}_2) (\hat{s} \times \hat{s}_2) + \hat{s}_2 (\hat{s} \cdot \hat{s}_2) [\hat{s}_2 (\hat{s} \cdot \hat{s}_2)] - \hat{s} \hat{s}_2 (\hat{s} \cdot \hat{s}_2) \right\}. \tag{19}
\]

Whereas \( r_j \) is measured with respect to the center of the DOE, \( r_2 \) is centered on the objective lens’ input pupil and \( r \) is centered on the optical train’s focal point. These choices eliminate additional phase factors in the Debye-Wolf integrals.

Equations (12) through (19) describe the superposition of plane waves projected by the objective lens onto the sample arising from each pixel in the DOE. The total field includes the incident field from Eq. (18) and also the field scattered by the sample itself.

**LIGHT SCATTERING BY SMALL OBJECTS**

Although light scattering by small objects has been studied in great detail, analytic results are available only for a few specific cases. Lorenz-Mie theory provides exact expansions in vector spherical harmonics for the scattering of plane waves by spheres. More general systems can be treated with T-matrix theory [23].

From Eq. (18), the object is illuminated by a superposition of plane waves. The component propagating along \( \hat{s} \) with polarization \( \mu \) has the complex amplitude \( A^I_{\mu}(\hat{s}) \). This component contributes \( A^S_{\mu}(r, \hat{s}) \) to the scattered field at point \( r \), where the subscript \( \mu \) refers to the polarization of the incident light. The total scattered field at \( r \) is then

\[
A^S(r) = \sum_{\mu=1}^2 \int_{\Omega} A^S_{\mu}(r, \hat{s}) \, d\Omega \tag{20}
\]

and the total vector potential is

\[
A(r) = A^I(r) + A^S(r). \tag{21}
\]

This can be substituted into Eqs. (2) through (4) to obtain the optical force and torque on the object due to the particular hologram implemented with the DOE.

In many cases of practical interest, the object may be modeled as a smooth sphere made of a homogeneous isotropic material with complex refractive index \( n_p \). The imaginary part of \( n_p \), also known as the extinction coefficient, characterizes the material’s absorptivity. Although absorption plays a critical role in computations of optically-induced torque [24], its influence on the forces exerted by optical traps has received less attention [25–27].
Illuminating such a sphere with $A_1^1(r, \hat{z})$, which describes a plane wave propagating along $\hat{z}$ and linearly polarized along $\hat{x}$, gives rise to a scattered field

$$A^S(r, \hat{z}) = \mathbb{LM}_1(r) A_1^1(r, \hat{z}),$$

(22)

where the Lorenz-Mie scattering tensor for $\hat{x}$-polarized light is given [28, 29] as an expansion,

$$\mathbb{LM}_1(r) = \sum_{n=1}^{\infty} \left( a_n N_{eln}^{(3)}(r) - b_n M_{eln}^{(3)}(r) \right),$$

(23)

in the vector spherical harmonics

$$M_{eln}^{(3)}(r) = \frac{\cos \phi}{\sin \theta} P_n^1(\cos \theta) \frac{j_n(kr)}{kr} \hat{r}$$

\[ - \sin \phi \frac{dP_n^1(\cos \theta)}{d\theta} \frac{j_n(kr)}{kr} \hat{\phi} \] 

(24)

and

$$N_{eln}^{(3)}(r) = n(n+1) \cos \phi P_n^1(\cos \theta) \frac{j_n(kr)}{kr} \hat{r}$$

\[ + \cos \phi \frac{dP_n^1(\cos \theta)}{d\theta} \frac{1}{kr} \frac{d}{dr} [r j_n(kr)] \hat{\theta} 
\]

\[ - \sin \phi \frac{1}{\sin \theta} P_n^1(\cos \theta) \frac{1}{kr} \frac{d}{dr} [r j_n(kr)] \hat{\phi} \] 

(25)

Here, $P_n^1(\cos \theta)$ is the associated Legendre polynomial of the first kind, and $j_n(kr)$ is the spherical Bessel function of the first kind of order $n$. The expansion coefficients in Eq. (23) are given by [28]

$$a_n = \frac{m^2 j_n(mx) \left[ x j_n(x) \right] - j_n(x) \left[ mx j_n(mx) \right]'}{m^2 j_n(mx) \left[ x h_n^{(1)}(x) \right] - h_n^{(1)}(x) \left[ mx j_n(mx) \right]'},$$

(26)

$$b_n = \frac{j_n(mx) \left[ x j_n(x) \right] - j_n(x) \left[ mx j_n(mx) \right]'}{j_n(mx) \left[ x h_n^{(1)}(x) \right] - h_n^{(1)}(x) \left[ mx j_n(mx) \right]'},$$

(27)

where $m = n_p/n_m$ is the particle’s relative refractive index relative, $x = ka$ is its size parameter, $h_n^{(1)}(x)$ is the spherical Hankel function of the first type of order $n$, and where primes denote derivatives with respect to the argument. Similarly,

Once these scattering coefficients are known, equivalent results for other incident directions, $\hat{s}$, and polarizations, $\mu$, can be obtained through coordinate rotations.

From the perspective of practical implementation, the sum in Eq. (23) converges after a number of terms, $n_c = x + 4.05 x^{1/3} + 2$, that depends on the particle’s size through $x$ [23, 29]. Computing additional terms does not improve accuracy because of the accumulation of round-off error. Making matters more difficult, the numerical implementations of the Bessel functions in at least some standard mathematical libraries suffer from large errors at large indexes $n$, and either large or small arguments $x$. The error can be assessed by computing the discontinuity in the total field just inside and just outside the particle’s surface. The implementations in recent versions of such commercial general purpose scientific computing packages as Mathematica, IDL and Matlab lead to relative discontinuities as large as 10 percent at the surface of a 100 nm diameter titania sphere in water. To avoid such errors, we employed the more robust numerical techniques described by [30] and [31] and typically obtain convergences to within $10^{-5}$ over the entire range of sizes considered.

**SUPERPOSITION OF PLANE-WAVE CONTRIBUTIONS**

The result analogous to Eq. (20) for scattering of light propagating along $\hat{z}$ and polarized in the $\hat{y}$ direction is obtained through a rotation of $\pi/2$ about $\hat{z}$: $\mathbb{LM}_2(r, \theta, \phi) = \mathbb{LM}_1(r, \theta, \phi - \pi/2)$. The scattered wave due to a plane wave incident along $\hat{s} = (\theta_s, \phi_s)$ has the same form in the coordinate system, $r' = (x', y', z')$, that is rotated so that $\hat{s}$ is aligned with $\hat{z}'$. The necessary coordinate transformation can be performed with the Euler rotation tensor

$$E(\hat{s}) = \begin{pmatrix}
(\cos \theta_s - 1) \cos^2 \phi_s + 1 & \sin \phi_s \cos \phi_s (\cos \theta_s - 1) & -\sin \theta_s \cos \phi_s \\
\sin \phi_s \cos \phi_s (\cos \theta_s - 1) & \sin^2 \phi_s (\cos \theta_s - 1) + 1 & -\sin \theta_s \sin \phi_s \\
\sin \theta_s \cos \phi_s & -\sin^2 \phi_s (\cos \theta_s - 1) + 1 & \cos \theta_s
\end{pmatrix},$$

(28)

such that $r' = E(\hat{s}) r$. The general solution for the scattered wave therefore has the form

$$A^S(r, \hat{s}) = E^{-1}(\hat{s}) \mathbb{LM} (E(\hat{s}) r) \mathbb{E}(\hat{s}) A^1(\hat{s}).$$

(29)

This can be combined with Eq. (18) for the incident field’s vector potential and substituted into Eqs. (1) through (4) to obtain the force and torque acting on the
FIG. 2: (a) Magnitude $Q_{\text{max}}$ and (b) range $r_{\text{max}}$ of the axial trapping force for a sphere of radius $a$ and relative refractive index $n_p/n_m$ in an optical tweezer created from light of wavenumber $k$. The solid curve denotes marginal trapping conditions with $Q_{\text{max}} = 0$. The dashed curve in (b) indicates conditions for which the trap’s axial well extends 1 µm.

sphere.

NUMERICAL RESULTS

To illustrate the vectorial theory of holographic optical trapping outlined above, we have computed the force and torque fields associated with several types of optical traps acting on colloidal spheres.

Trapping by an optical tweezer

Following Ref. [32], we begin by computing the optical force experienced by a colloidal sphere in a conventional point-like optical tweezer as a function of the sphere’s radius, $a$, and its refractive index $n_p$. The incident light field is obtained by setting $\varphi_j = 0$ for all of the pixels in the DOE. We focus our attention on the axial force profile, which tends to be weaker than the longitudinal force profile. Failure to achieve axial trapping is the principal failure mode of conventional optical tweezers.

The results, summarized in Fig. 2, are similar in many respects to those reported in Ref. [32]. Most tellingly, Fig. 2(a) and Fig. 3 of [32] both demonstrate that particles with diameters larger than $\lambda/2$ can be trapped only if their relative refractive index is below roughly 1.4. The large domains of high-index trapping in [32] are reduced to discrete islands of stability in Fig. 2 due in part to the influence of the relay optics on the light’s polarization, which was not considered in [32].

Both sets of results predict that high-index spheres can be trapped in a single-beam optical tweezer provided they are sufficiently small. This is an important observation for holographic assembly of photonic structures [33], many of which rely on high-index materials for their interesting and useful optical properties.

Figure 3 of [32] suggests that spheres smaller than roughly $\lambda/4$ cannot be trapped at all, and that the condition for marginal trapping depends strongly on refractive index for small index mismatches. This differs qualitatively from Fig. 2, which shows stable trapping for very small spheres, even with modest relative refractive indexes. The difference in this case can be ascribed to the earlier study’s use of Matlab’s spherical Hankel functions, which are inaccurate for large indexes and small arguments. Consequently, Fig. 2 should be considered a more faithful guide for designing optical trapping experiments.

Forces and torques in an optical vortex

An optical tweezer may be transformed into a torque-exerting optical vortex [34–36] by imposing the helical wavefront phase profile

$$\varphi_j = \ell \theta_j \text{ mod } 2\pi$$

with the DOE. The winding number $\ell$ controls the beam’s helicity and is referred to as the topological charge. The helical topology gives rise to destructive interference along the beam’s axis. Light therefore is redirected to a ring of radius $R_\ell$ that is proportional to $\ell$ in typical holographic implementations [37–39]. The image in Fig. 3(a) shows the computed intensity in the focal plane, $z = 0$, for half of such a ring. In this case, the topological charge $\ell = 60$ corresponds to the radius $R_\ell = 5 \mu m$.

Figure 3(b) shows the in-plane component of the total optical force $\mathbf{F}(r)$ for a sphere with $ka = 5.9$, $n_p = 1.46 + 10^{-5}i$, and $n_m = 1.33$. These values are appropriate for a 1 µm diameter silica sphere dispersed in water and trapped at $\lambda = 532$ nm. Hues indicate the direction of the force in the ($x, y$) plane according to the inset color wheel. The saturation of the color corresponds to the magnitude of the force, with unsaturated
white regions corresponding to weak forces, and brightly saturated regions corresponding to $Q_{\text{max}} = 0.007$. The six black dots in Fig. 3(b) show the starting points for the computed trajectories that are superimposed on the force field. These two-dimensional trajectories are calculated for particles constrained to move in the $(x, y)$ plane, and do not account for the axial component of the force. They correspond to the motion typically described in experimental studies of colloidal particles in high-index optical vortexes [37, 40, 41], where particles are pressed against a glass surface to prevent them from escaping along the axial direction. These representative trajectories show that particles are drawn by intensity-gradient forces to the bright ring and then are driven around the ring by phase-gradient forces [7, 42]. This circulatory motion is a consequence of the helical beam’s orbital angular momentum, which amounts to $\ell \hbar$ per photon [43, 44]. The orbital angular momentum flux in a helical mode is independent of the photons’ spin, and thus is independent of the light’s polarization [43, 45].

Although the incident laser beam is assumed to be linearly polarized, the strongly focused light field has a far more complicated spatially varying polarization. Gradients in the intensity, phase and polarization of the light can exert torques as well as forces on illuminated objects, as the in-plane torque distribution in Fig. 3(c) demonstrates. The hue in Fig. 4(c) indicates direction of the torque in the $(x, y)$ plane, and the saturation indicates the magnitude of the torque efficiency. A homogeneous isotropic sphere only experiences a torque if it absorbs light [24]. The scale of the torque efficiency in Fig. 3(c) is therefore proportional to the imaginary part of $n_P$. For the micrometer-diameter silica sphere in this calculation, $\tau_{\text{max}} = 2 \times 10^{-6}$. The maximum rotation frequency of $0.1 \text{ Hz/W}$ would be challenging to observe experimentally, particularly on a background of vigorous brownian motion. Nevertheless, this demonstrates that optically-induced rotation can arise even in linearly polarized optical traps, and may become an important factor for materials such as polystyrene that absorb light more strongly than silica.

The axial structure of an optical vortex, plotted in Figs. 3(d), (e) and (f), reveals its limitations as an optical trap. The axial intensity profile in Fig. 3(d) corresponds to a region around the principal focal ring, the dashed line indicating the position of the focal plane, $z = 0$. The associated force distribution in Fig. 3(e) shows that particles are driven downstream along the optical axis, and so are not axially trapped. Hues correspond to directions in the $(x, z)$ plane corresponding to the color wheel, and maximum saturation corresponds to the force scale $Q_z = 0.006$. 

The six black dots in Fig. 3(b) show the starting points for the computed trajectories that are superimposed on the force field. These two-dimensional trajectories are calculated for particles constrained to move in the $(x, y)$ plane, and do not account for the axial component of the force. They correspond to the motion typically described in experimental studies of colloidal particles in high-index optical vortexes [37, 40, 41], where particles are pressed against a glass surface to prevent them from escaping along the axial direction. These representative trajectories show that particles are drawn by intensity-gradient forces to the bright ring and then are driven around the ring by phase-gradient forces [7, 42]. This circulatory motion is a consequence of the helical beam’s orbital angular momentum, which amounts to $\ell \hbar$ per photon [43, 44]. The orbital angular momentum flux in a helical mode is independent of the photons’ spin, and thus is independent of the light’s polarization [43, 45].

Although the incident laser beam is assumed to be linearly polarized, the strongly focused light field has a far more complicated spatially varying polarization. Gradients in the intensity, phase and polarization of the light can exert torques as well as forces on illuminated objects, as the in-plane torque distribution in Fig. 3(c) demonstrates. The hue in Fig. 4(c) indicates direction of the torque in the $(x, y)$ plane, and the saturation indicates the magnitude of the torque efficiency. A homogeneous isotropic sphere only experiences a torque if it absorbs light [24]. The scale of the torque efficiency in Fig. 3(c) therefore is proportional to the imaginary part of $n_P$. For the micrometer-diameter silica sphere in this calculation, $\tau_{\text{max}} = 2 \times 10^{-6}$. The maximum rotation frequency of $0.1 \text{ Hz/W}$ would be challenging to observe experimentally, particularly on a background of vigorous brownian motion. Nevertheless, this demonstrates that optically-induced rotation can arise even in linearly polarized optical traps, and may become an important factor for materials such as polystyrene that absorb light more strongly than silica.
Trapping in a holographic ring trap

The data in Fig. 4 show the corresponding results for a micrometer-diameter silica sphere in a holographic ring trap [46]. Unlike optical vortexes, these ring traps are designed to come to a diffraction-limited focus. The in-plane and axial intensity distributions in Figs. 4(a) and (d) demonstrate ring traps’ superior focusing characteristics. Holographic ring traps also may have arbitrary circumferential phase profiles. The example in Fig. 4 was created with a uniform azimuthal phase gradient corresponding to \( \ell = 30 \). Such a ring trap can be created holographically with [46]

\[
a_j = \left| J_\ell \left( \frac{kRr_j}{2f} \right) \right| \quad \text{and} \quad \phi_j = \left[ \ell \theta_j + \pi H \left( -J_\ell \left( \frac{kRr_j}{2f} \right) \right) \right] \mod 2\pi ,
\]

where \( J_\ell(x) \) is the \( \ell \)-th order Bessel function of the first kind and \( H(x) \) is the Heaviside step function. In practice, we have encoded this complex-valued hologram on a phase-only spatial light modulator using the shape-phase algorithm [46]. Whereas an optical vortex's radius is coupled to its helicity, Eq. (31) shows that the radius \( R \) of a ring trap is independent of \( \ell \).

The ring trap’s in-plane characteristics, plotted in Figs. 4(b) and (c), qualitatively resemble those of the optical vortex. The radial force scale is substantially increased, however, by the stronger intensity gradients, with \( Q_{\text{max}} = 0.048 \). This also is reflected in the larger torque scale in Fig. 4(c), with \( \tau_{\text{max}} = 2 \times 10^{-5} \).

The most noteworthy characteristic of the holographic ring trap is its axial force profile, which is shown in Fig. 4(e). Unlike the optical vortex, the ring trap features strong axial focusing and so has a stable axial equilibrium point just downstream of the focal plane with a peak trapping efficiency of \( Q_{\text{max}} = 0.042 \). Particles thus can be trapped in three dimensions as they circulate around the ring, without requiring additional external confinement. This should improve the performance of ring-based micro-optomechanical machines [47] and should facilitate a search for spin induced by the optical torques shown in Figs. 4(c) and (f).

Holographic line trap

Figure 5 shows the intensity, force and torque distribution experienced by a micrometer-diameter silica sphere in a holographic line trap [20]. This is another generalization of an optical tweezer whose focal point is extended along a line segment in the focal plane, as shown in Fig. 5(a). Holographic line traps also come to a diffraction-limited focus in the axial direction, as Fig. 5(d) shows, and therefore can trap objects stably in three dimensions, as shown in Fig. 5(e), with a trapping efficiency of \( Q_{\text{max}} = 0.05 \). The addition of an appropriate phase profile then facilitates creating a tailored force profile along the line’s length [7], even when its intensity is uniform. The effect of a confining phase profile is demonstrated in Fig. 5(b), with superimposed trajectories converging on a region of mechanical equilibrium [7]. A uniform phase profile eliminates the inward force along the line, and would allow a particle to diffuse freely in the \( \hat{y} \) direction [7]. Switching the sign of the phase profile would drive particles to the ends of the line [7].

The computed torque distributions in Figs. 5(c) and (f) show that an illuminated particle again would tend to spin in the proximity of the strongly focused trap. The torque distribution for a holographic line trap is simpler than that for a holographic ring trap, with the particle’s rotation axis varying little along the line and its direction flipping as the particle crosses the line. The torque efficiencies are comparably large, however.

CONCLUSION

We have combined a Debye-Wolf treatment of light propagation through an optical train with Lorenz-Mie theory for light scattering to develop a vectorial theory for the forces and torques applied by holographic optical traps. This theory of holographic trapping is inherently more accurate than approximations based on scalar
diffraction theory or on the Rayleigh or ray-optics approximations. It not only reproduces previous results obtained for optical tweezers and related single-beam optical traps, but accurately accounts for polarization effects in a realistic model for the optical train of practical holographic trapping systems. Treating the hologram on a pixel-by-pixel basis not only permits detailed analysis of the very general optical force fields that can be projected holographically, but also lends itself to accurate treatment of aberrations, which can be encoded in the DOE.

We have used this vector theory to confirm the behavior of phase-gradient forces in extended traps that was predicted on the basis of scalar diffraction theory [7]. The spatially resolved images of the force and torque fields further reveal substantial consequences of polarization rotation in high-numerical-aperture optics. The transformation of the linearly polarized input beam into more general and spatially varying elliptical polarization can give rise to a highly structured torque field whose behavior of phase-gradient forces in extended traps that was predicted on the basis of scalar diffraction theory [7]. The spatially resolved images of the force and torque fields further reveal substantial consequences of polarization rotation in high-numerical-aperture optics. The transformation of the linearly polarized input beam into more general and spatially varying elliptical polarization can give rise to a highly structured torque field whose behavior of phase-gradient forces in extended traps that was predicted on the basis of scalar diffraction theory [7].

This work was supported in part by the NSF through Grant Number DMR-0606415 and in part by a grant from the Keck Foundation. B.S. acknowledges support from a Kessler Family Foundation Fellowship.

[43] L. Allen, M. W. Beijersbergen, R. J. C. Spreeuw, and


