

Modified Effective-Range Theory

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ABSTRACT

Developments in effective-range theory as applied to low-energy atomic scattering processes are reviewed. The focus is on the seminal contributions to this theory made by Larry Spruch, who showed how its original form could be extended to account for the presence of long-range forces arising from Coulomb interactions. The influence of this body of work on the very recent developments stimulated by advances in experimental techniques is emphasized.

Some time ago, Wigner [1], in the context of the theory of nuclear reactions, formulated a law governing the energy dependence of transition amplitudes for energies near the threshold for the opening of a new reaction channel. The significance of the law lies in its generality; the threshold law is independent of the detailed dynamics of the reaction other than the short-range nature of the nuclear interaction. It was then seen [2] that with the threshold behavior accounted for, the amplitude for nucleon-nucleon scattering could be represented as a linear function of the scattering energy, in terms of two parameters, referred to as the scattering length and effective range. Here again, the significance of this “effective-range” theory lies in its generality; one requires only that the nuclear potential be of short range. In particular, it became apparent that an observation of low-energy neutron-proton scattering could provide information on, say, the depth and range of the potential but not on any of its finer details. A typical application may involve the extrapolation of scattering data based on measurements performed at a few different energies. Bound-state properties may be probed as well. Suitably extended, the theory allows for the inclusion of a Coulomb interaction [3].

There was a resurgence of interest in atomic scattering processes in the early sixties, and it was recognized at the time by a number of people that some version of effective-range theory would be extremely valuable in the analysis of low-energy electron-atom scattering data. It was Spruch, however, who not only understood, on physical grounds, that the theory would have to be modified to account for the long-range force that a slowly moving electron experiences when incident on a neutral, polarizable atomic system, but who also understood how to develop such a modified effective-range theory. The dominant component of the long-range potential is proportional to the electric dipole polarizability of the atomic target, behaving at great distances as the inverse fourth power of the electron-atom separation. It was necessary to replace the sinusoidal solutions describing free-particle propagation that appear in standard scattering theory by solutions—these were known to be Mathieu functions [4]—accounting for the polarization potential. With this accomplished one had the expectation that the energy dependence for scattering, in the presence of the polarization potential, would show the smooth behavior allowing for a simple parametrization of the effective-range type. (Such a “distorted-wave” approach is similar in spirit to the standard treatment of nuclear scattering in the presence of a Coulomb potential.)

The theory developed by Spruch and coworkers [5, 6] took as its model the version of ordinary effective-range theory given by Bethe [2]. In that analysis four wave functions were introduced, two of them corresponding to scattering at zero energy and at a positive energy close to zero, with the remaining two wave functions derived from the asymptotic forms of the former pair (chosen with energy-independent normalization). Straightforward manipulation of the wave equations satisfied by these functions, along with the assumption that the energy dependence of the physical solution is weak in the region where the scattering potential is strong, leads directly to the desired effective-range expansion. Of course the form of the sinusoidal asymptotic solutions is known everywhere and, in particular, near the origin where it is required in the derivation. With the replacement of these free-particle wave functions by Mathieu functions one is faced with the problem of

finding connection formulas relating their behavior at great distances and near the origin. The theory was then worked out once this technical problem was solved. Interestingly, the development of connection formulas of this type is required in versions of “quantum defect theory”, a formalism that is currently widely used as the basis for atomic collision and structure calculations [7].

A novel feature of the modified effective-range expansion is the appearance of terms depending logarithmically on the energy variable. More significant is the modification of the Wigner threshold law for states of orbital angular momentum greater than zero. For each of these partial waves the low-energy phase shift is proportional to the energy, while the Wigner law would predict a higher power of the energy, increasing with higher angular momentum quantum numbers. Moreover, for a given energy and angular momentum, the polarization phase shift is found from a knowledge of the polarizability of the target, a parameter that can be determined by measurements that are independent of the scattering process [8]. The predictive nature of this feature of the theory has drawn considerable interest over the years; it permits, for example, the extrapolation of scattering data into the very low energy region not easily accessible experimentally. As a result, the theory has become a standard part of textbook discussions of atomic scattering processes.

It is natural to extend the study of the effect of long-range forces to electron-ion scattering. In one of the early developments of multichannel quantum defect theory, Gailitis [9] considered the scattering of an electron by a positive hydrogenic ion near the threshold for excitation of the target to a state of higher angular momentum. Owing to their degeneracy, such states do not have well-defined parity and this results in the appearance of an inverse-square component to the effective potential resulting from the interaction of the incident electron with the dipole field of the excited ion. In this problem the distorted waves could be determined analytically through a modification of the orbital quantum number of the undistorted Coulomb waves. In the more general case of electron scattering by a charged, polarizable target, inclusion into quantum defect theory (using lowest-order

perturbation theory) of the effect of the polarization potential [10], as well as a wider class of multipole potentials [11], has been considered. A different approach was taken by Spruch, based on the idea that it is perfectly sensible, in studying a multiparticle scattering problem, to adopt numerical methods in solving any auxiliary one-body problem that is encountered. The standard Chew-Goldberger treatment [3] was extended with the aid of a numerical treatment of the effect of the polarization force, making it possible to extract out exactly the strongly energy-dependent terms [12]. Results were later tabulated in a convenient form [13].

A study of the formal expression for the effective electron-atom potential allows for an examination of components of longer range than the polarization potential. The coefficient of the inverse sixth-power potential was determined by Spruch and collaborators [14], and independently by others [15, 16]. Interest then settled on a possible extension of modified effective-range theory to include the effect of this longer-range component. A more flexible approach was required for this purpose since analytic expressions for the asymptotic wave functions appropriate to this more complicated potential are not available. A version of the distorted-wave Born approximation was formulated by Spruch to deal with this problem [17]. The theory has an attractive simplicity, requiring only the evaluation of standard integrals, with no need to rely on detailed properties of special functions. In addition to uncovering the new terms in the expansion accounting for the extended form of the long-range potential, this work allowed for applications to a wider class of atomic reactions, including photoexcitation.

A different procedure was developed subsequently [18], in which the effect of a superposition of long-range potentials on the form of the asymptotic states is determined with the aid of a novel perturbation theory, one involving the introduction of basis states corresponding to nonintegral values of the orbital quantum number. Multichannel extensions have been developed [19]. The dominant long-range component of the atom-atom effective potential is, of course, the van der Waals inverse sixth-power potential. An

analytic treatment of the modification of effective-range theory due solely to this potential has recently been worked out within the context of quantum defect theory [20]. For electron scattering by an atom with a permanent quadrupole moment the effective potential has an inverse-cube component in addition to contributions of longer range. The near-threshold energy dependence of the full scattering matrix for this class of problems was determined by O'Malley [21], who extended the asymptotic expansion approach of Levy and Keller [22]. This extension was based on a three-dimensional version, developed earlier by Spruch [23], of the "variable phase" method. It seems fair to say that the influence of Spruch's early work is clearly evident in ongoing efforts in this area of atomic theory.

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